

Equivalence relations that act on bundles of hyperbolic spaces

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Abstract

Consider a measured equivalence relation acting on a bundle of hyperbolic metric spaces by isometries. We prove that every aperiodic hyperfinite subequivalence relation is contained in a *unique* maximal hyperfinite subequivalence relation. We classify elements of the full group according to their action on fields on boundary measures (extending earlier results of Kaimanovich), study the existence and residuality of different types of elements and obtain an analogue of Tits' alternative.

Keywords: measured equivalence relations, Tits alternative, treeable equivalence relations
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1 Introduction

The broad goal of this research is to generalize the theory of groups acting on hyperbolic spaces to measured equivalence relations. To explain this properly we introduce some notation, leaving details to later sections.

Let (X, μ) denote a standard probability space and $\mathcal{R} \subset X \times X$ a discrete Borel equivalence relation. We require that μ is \mathcal{R} -invariant which means that if $\phi : X \rightarrow X$ is any Borel isomorphism with graph contained in \mathcal{R} , then $\phi_*\mu = \mu$. We let $[\mathcal{R}]$, the **full group**, denote the set of all such Borel isomorphisms up to equivalence (isomorphisms are equivalent if they agree μ -almost everywhere). We usually require that \mathcal{R} is ergodic which means that if $Y \subset X$ is any Borel set equal to a union of \mathcal{R} -classes then $\mu(Y) \in \{0, 1\}$. The triple (X, μ, \mathcal{R}) is an ergodic **discrete probability-measure-preserving (pmp) equivalence relation**.

Definition 1 (metric and Borel bundles). A **Borel bundle over** X is a standard Borel space B with a Borel surjection $\pi : B \rightarrow X$ called the **bundle projection**. We let $B * B = \{(y, z) : \pi(y) = \pi(z)\}$ denote the fiber product with its Borel structure inherited from the inclusion $B * B \subset B \times B$. For each $x \in X$, the **fiber** over x is the subset $\pi^{-1}(x)$. It is denoted by $B_x := \pi^{-1}(x)$. A **metric bundle over** X consists of a Borel bundle $\pi : B \rightarrow X$ with a Borel map $d : B * B \rightarrow [0, \infty)$ such that for each $x \in X$, B_x equipped with the restriction $d \upharpoonright B_x \times B_x$ is a metric space. A **section** is a map $\sigma : X \rightarrow B$ such that $\sigma(x) \in B_x$ for all x . A metric bundle is **separable** if there exists a countable set $\{\sigma_i\}_{i \in \mathbb{N}}$ of Borel sections $\sigma_i : X \rightarrow B$ such that for every $x \in X$, the set $\{\sigma_i(x)\}_{i \in \mathbb{N}}$ is dense in B_x . If

each $(B_x, d \upharpoonright B_x \times B_x)$ is a geodesic Gromov hyperbolic space (the definition of which is in §A below) then we say B is a **bundle of hyperbolic spaces**.

Definition 2. Let $\pi : \mathcal{H} \rightarrow X$ denote a bundle of hyperbolic spaces. An **action of the equivalence relation \mathcal{R} on \mathcal{H} by isometries** consists of a family $\{\alpha(x, y) : (x, y) \in \mathcal{R}\}$ of isometries $\alpha(x, y) : \mathcal{H}_y \rightarrow \mathcal{H}_x$ satisfying:

- (cocycle condition) $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$ for all $x\mathcal{R}y\mathcal{R}z$,
- (Borel condition) $\{(p, q) \in \mathcal{H} \times \mathcal{H} : \alpha(\pi(p), \pi(q))(q) = p\}$ is Borel.

The tuple $(\mathcal{H}, d, \pi, \alpha)$ is an **isometric action of \mathcal{R} on a bundle of hyperbolic spaces**.

Example 1. A **graphing** is a Borel subset $\mathcal{G} \subset \mathcal{R}$ such that \mathcal{R} is the smallest equivalence relation containing \mathcal{G} and \mathcal{G} is symmetric (which means $(x, y) \in \mathcal{G} \Rightarrow (y, x) \in \mathcal{G}$). For each $x \in X$, we let \mathcal{G}_x denote the **graph at x** . It has vertex set $[x]_{\mathcal{R}}$ (the equivalence class of x) and edge set $E_x = \{\{y, z\} : (y, z) \in \mathcal{G} \text{ and } y, z \in [x]_{\mathcal{R}}\}$. By abuse of notation we also consider \mathcal{G}_x to be a metric graph by assigning each edge length 1. We say \mathcal{G} is **hyperbolic** if each \mathcal{G}_x is a Gromov hyperbolic metric space. In this case, let $\mathcal{H} = \sqcup_x \mathcal{G}_x$ be the disjoint union of the metric graphs. We consider \mathcal{H} to be a bundle with projection map that takes \mathcal{G}_x to x . It is a bundle of hyperbolic spaces where $d : \mathcal{H} * \mathcal{H} \rightarrow [0, \infty)$ is defined by setting $d(y, z)$ equal to the length of the shortest path from y to z in \mathcal{G}_x (if $\pi(y) = \pi(z) = x$). We define the action by setting $\alpha(x, y) : \mathcal{G}_y \rightarrow \mathcal{G}_x$ equal to the natural identification. Equivalence relations with hyperbolic graphings were studied by Kaimanovich in [Kai04].

Remark 1. We prefer to work with isometric actions on bundles of hyperbolic spaces instead of hyperbolic graphings for the following reason: the class of equivalence relations that admit such actions is closed under taking subequivalence relations (because we can always restrict the action to the subequivalence relation). By contrast, it is unknown whether the existence of a hyperbolic graphing is closed under taking subequivalence relations.

Definition 3 (Bundle of Gromov completions). Let $(\mathcal{H}, d, \pi, \alpha)$ be as in Definition 2. For each $x \in X$, let $\partial\mathcal{H}_x$ denote the Gromov boundary of \mathcal{H}_x and $\overline{\mathcal{H}}_x = \mathcal{H}_x \cup \partial\mathcal{H}_x$ denote the Gromov completion. We also let

$$\overline{\mathcal{H}} = \sqcup_x \overline{\mathcal{H}}_x, \quad \partial\mathcal{H} = \sqcup_x \partial\mathcal{H}_x \subset \overline{\mathcal{H}}$$

be the disjoint unions. We extend the projection map π to $\overline{\mathcal{H}}$ so that π maps $\overline{\mathcal{H}}_x$ to x (for $x \in X$). In this manner, we consider $\overline{\mathcal{H}}$ and $\partial\mathcal{H}$ to be bundles over X . In §2 we show that $\overline{\mathcal{H}}$ and $\partial\mathcal{H}$ are naturally endowed with Borel structures so that the inclusions $\mathcal{H} \rightarrow \overline{\mathcal{H}}, \partial\mathcal{H} \rightarrow \overline{\mathcal{H}}$ are Borel and the projection map $\pi : \overline{\mathcal{H}} \rightarrow X$ is Borel. We also extend the action α as follows. Because $\alpha(x, y) : \mathcal{H}_y \rightarrow \mathcal{H}_x$ is an isometry there is a unique extension, which we also denote by $\alpha(x, y) : \overline{\mathcal{H}}_y \rightarrow \overline{\mathcal{H}}_x$ that is a homeomorphism. It satisfies the cocycle condition and the Borel condition of Definition 2. These statements are proven in §2 below.

Definition 4 (Sections, orbits and limit sets). Now let $(\mathcal{H}, d, \pi, \alpha)$ be as in Definition 2. Recall that a section is a map $\sigma : X \rightarrow \mathcal{H}$ such that $\sigma(x) \in \mathcal{H}_x$ for a.e. x . This induces a map

$$\sigma_x : [x]_{\mathcal{R}} \rightarrow \mathcal{H}_x, \quad \sigma_x(y) := \alpha(x, y)\sigma(y).$$

The image of σ_x is the **orbit of x** , denoted \mathcal{O}_x^σ . In order to ensure non-triviality, we require that σ is **metrically proper** which means: for every metric ball $\mathcal{B} \subset \mathcal{H}_x$, $\mathcal{O}_x^\sigma \cap \mathcal{B}$ is finite. Let $\mathcal{L}_x^\sigma(\mathcal{R}) := \overline{\mathcal{O}_x^\sigma} \cap \partial\mathcal{H}_x$ be the **limit set** (where the closure of \mathcal{O}_x^σ is taken in $\overline{\mathcal{H}}_x$).

Many arguments rely on a map $d_\sigma : \mathcal{R} \rightarrow [0, \infty)$ defined by

$$d_\sigma(x, y) = d(\sigma_z(x), \sigma_z(y))$$

for any $z \in [x]_{\mathcal{R}} = [y]_{\mathcal{R}}$. The cocycle property of the action implies that this definition of d_σ is independent of the choice of z . This gives a pseudo-metric on each equivalence class $[x]_{\mathcal{R}}$. It is a metric if $\sigma_x : [x]_{\mathcal{R}} \rightarrow \mathcal{H}_x$ is injective. In this case σ_x is an isometric embedding of $[x]_{\mathcal{R}}$ into \mathcal{H}_x .

Our main results hold under the following hypotheses:

Definition 5 (Main Assumption). We say the **Main Assumption** is satisfied if (X, μ) is a standard non-atomic probability space, $\mathcal{R} \subset X \times X$ is a discrete Borel equivalence relation, μ is \mathcal{R} -invariant and ergodic, $\pi : \mathcal{H} \rightarrow X$ is a separable bundle of hyperbolic spaces with isometric action α , $\sigma : X \rightarrow \mathcal{H}$ is a metrically proper section and for a.e. x , the closure $\overline{\mathcal{O}_x^\sigma}$ of the orbit of x is compact in $\overline{\mathcal{H}}_x$. This compactness assumption is automatically satisfied if each fiber \mathcal{H}_x is locally compact.

Our first result is that the limit sets are essentially independent of σ :

Theorem 1.1. *If the Main Assumption is satisfied and $\sigma, \eta : X \rightarrow \mathcal{H}$ are metrically proper sections whose orbit closures $\overline{\mathcal{O}_x^\sigma}, \overline{\mathcal{O}_x^\eta}$ are compact (for a.e. x) then $\mathcal{L}_x^\sigma = \mathcal{L}_x^\eta$ for a.e. x .*

Because of this theorem, we may write $\mathcal{L}_x := \mathcal{L}_x^\sigma$.

1.1 Treeable subequivalence relations

Our main results concern the structure of subequivalence relations of \mathcal{R} . More precisely, they concern two specific types of subequivalence relations: treeable and hyperfinite. Let us recall that an equivalence relation \mathcal{R} is **treeable** if it admits a graphing \mathcal{G} (as in Example 1) in which each local graph \mathcal{G}_x is a tree. Such equivalence relations are analogous to free groups in group theory and have been studied intensively. For example treeings play a central role in [Gab00] (treeings realize the cost). A group is **treeable** if it admits an essentially free action whose orbit-equivalence relation is treeable. See [Gab05] for many examples of treeable groups. If \mathcal{R} is ergodic, treeable and non-hyperfinite then there exists an essentially free ergodic action $\mathbb{F}_2 \curvearrowright X$ of the rank 2 free group such that each orbit of the action is contained in an \mathcal{R} -class [GL09, Proposition 14]. Since trees are hyperbolic metric spaces, any treeable equivalence relation with finite cost satisfies the Main Assumption (with respect to the bundle defined in Example 1).

Let us also remark on the von Neumann-Day problem in group theory and its analog in the theory of equivalence relations. This problem asked whether every non-amenable group necessarily contains a non-amenable free group. It was disproven by Ol'shankii [Ol'91]. However, the analogous problem for equivalence relation, “does every non-hyperfinite pmp equivalence relation contain a non-hyperfinite treeable subequivalence relation?” remains open. A strong partial answer due to Gaboriau-Lyons [GL09] states that the orbit equivalence relation of any Bernoulli shift action (with large enough base entropy) of a non-amenable group has this property. Our first main result is a positive answer to this question under hyperbolicity assumptions:

Theorem 1.2. *If \mathcal{R} satisfies the Main Assumption and is non-hyperfinite then \mathcal{R} contains an ergodic non-hyperfinite treeable subequivalence relation.*

Because the hypotheses on \mathcal{R} are inherited by subequivalence relations, the result above also holds for all subequivalence relations of \mathcal{R} . Thus we may think of this as an analog of Tits' alternative for hyperbolic groups.

1.2 Hyperfinite subequivalence relations

Let us now recall that an equivalence relation \mathcal{R} is **hyperfinite** if there exist Borel subequivalence relations $\mathcal{R}_1 \leq \mathcal{R}_2 \leq \dots$ such that

- for each n , all \mathcal{R}_n -classes are finite;
- $\mathcal{R} = \bigcup_n \mathcal{R}_n$.

Hyperfiniteness is analogous to amenability in group theory. In fact, it is equivalent to amenability in the theory of equivalence relations [OW80, CFW81]. It is well-known that, in a hyperbolic group, any infinite amenable subgroup is contained in a unique maximal amenable subgroup (which must, in fact, be virtually cyclic). Moreover, normalizers of infinite amenable subgroups are necessarily amenable. Our next result shows that this phenomenon extends to equivalence relations:

Theorem 1.3. *If \mathcal{R} satisfies the Main Assumption and $\mathcal{S} \leq \mathcal{R}$ is an aperiodic hyperfinite subequivalence relation then \mathcal{S} is contained in a unique maximal hyperfinite subequivalence relation. Moreover, the subequivalence relation generated by \mathcal{S} and the normalizer of $[\mathcal{S}]$ in $[\mathcal{R}]$ is hyperfinite.*

Remark 2. It is easy to prove that every hyperfinite subequivalence relation is contained in some maximal hyperfinite subequivalence relation. However, uniqueness is not true in general. For example let G be any countable group with infinite amenable groups H_1, H_2, H_3 such that $\langle H_1, H_2 \rangle, \langle H_2, H_3 \rangle$ are amenable but $\langle H_1, H_2, H_3 \rangle$ is non-amenable (for example this property is satisfied by $G = SL(3, \mathbb{Z})$ with respect to its elementary subgroups). If \mathcal{R} denotes the orbit equivalence relation of an essentially free pmp $G \curvearrowright (X, \mu)$ and $\mathcal{S}_{ij} \leq \mathcal{R}$ is the subequivalence relation generated by the $\langle H_i, H_j \rangle$ -action then $\mathcal{S}_{12}, \mathcal{S}_{23}$ and $\mathcal{S}_2 = \mathcal{S}_{12} \cap \mathcal{S}_{23}$ are aperiodic and hyperfinite. But since the subequivalence relation generated by \mathcal{S}_{12} and \mathcal{S}_{23} is non-hyperfinite, there is more than one maximal hyperfinite subequivalence relation containing \mathcal{S}_2 .

Remark 3. The von Neumann algebra analog of Theorem 1.3 is open even in the case of free group factors (see the end of [PT11] where this is stated as a conjecture). There are several constructions of maximal amenable subalgebras of von Neumann algebras with hyperbolic flavor [CFRW10, Hou14, She06]. Recently (and independently of this research) R. Boutonnet and A. Carderi [BC14] have shown that if $H < G$ is a maximal amenable subgroup of a word hyperbolic group and $G \curvearrowright (X, \mu)$ any essentially free pmp action then the orbit equivalence relation of the H -action is a maximal hyperfinite subequivalence relation of the orbit-equivalence relation the G -action.

1.2.1 Parabolic and loxodromic subequivalence relations

We now turn towards a more detailed picture of the hyperfinite subequivalence relations of \mathcal{R} analogous to the elliptic/parabolic/loxodromic classification of isometries of real hyperbolic space. This classification arises from considering the action of \mathcal{R} on fields of boundary measures. To be precise, a **field of boundary measures** is an assignment $x \mapsto \nu_x$ where ν_x is a Borel probability measure on $\partial\mathcal{H}_x$ and the assignment satisfies a certain Borel condition (see §C). We do not distinguish between fields that agree almost everywhere. The space $\text{Prob}(\partial\mathcal{H} \rightarrow X)$ of all Borel fields of boundary measures admits the structure of a compact convex subspace of a Banach space (Lemma C.2). Moreover, the full group $[\mathcal{R}]$ acts jointly continuously on $\text{Prob}(\partial\mathcal{H} \rightarrow X)$ by $(f\nu)_{fx} = \alpha(fx, x)_*\nu_x$. The subspace $\text{Prob}(\mathcal{L} \rightarrow X) \subset \text{Prob}(\partial\mathcal{H} \rightarrow X)$ of fields with ν_x supported on the limit set \mathcal{L}_x is a minimal set for this action (Corollary 7.3) whenever \mathcal{R} is non-hyperfinite. By contrast, if \mathcal{R} is hyperfinite then $[\mathcal{R}]$ is extremely amenable [GP07] and therefore every minimal action is trivial.

Given a subequivalence relation $\mathcal{S} \leq \mathcal{R}$, let $\text{Fix}(\mathcal{S})$ be the set of $\eta \in \text{Prob}(\partial\mathcal{H} \rightarrow X)$ that are fixed by \mathcal{S} (so $\alpha(x, y)_*\eta_y = \eta_x$ for $(x, y) \in \mathcal{S}$). It is essentially a result of Kaimanovich [Kai04] that $\text{Fix}(\mathcal{S})$ is nonempty if and only if \mathcal{S} is hyperfinite. Moreover, if \mathcal{S} is aperiodic then for any $\eta \in \text{Fix}(\mathcal{S})$ the support of η_x has cardinality at most 2. So we say \mathcal{S} is

- **parabolic** if there is a unique $\eta \in \text{Fix}(\mathcal{S})$ and for a.e. x , the support of η_x has cardinality 1;
- **loxodromic** if there exists $\eta \in \text{Fix}(\mathcal{S})$ such that for a.e. x , the support of η_x has cardinality 2;

- **mixed** if there is a nontrivial disjoint measurable partition $X = Y \sqcup Z$ such that $\mathcal{S} \upharpoonright Y$ is parabolic and $\mathcal{S} \upharpoonright Z$ is loxodromic where \upharpoonright denotes “restricted to”. Moreover this partition is unique up to null sets.

It follows from Kaimanovich’s results that if $\mathcal{S} \leq \mathcal{R}$ is hyperfinite and aperiodic then it is either parabolic, loxodromic or mixed. Moreover, if \mathcal{S} is ergodic then it must be either parabolic or loxodromic. Similarly, if $f \in [\mathcal{R}]$ then we say that f is parabolic/loxodromic/mixed if the subequivalence relation generated by f is parabolic/loxodromic/mixed. Let $\text{APER}, \text{PARA}, \text{LOXO} \subset [\mathcal{R}]$ denote the subsets of aperiodic, parabolic and loxodromic elements. As shown in [Kec10, I.2], APER is a G_δ subset of $[\mathcal{R}]$; in particular, it is a Polish space. We prove

Theorem 1.4. *If the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite then both PARA and LOXO are nonempty. Moreover, PARA is a dense G_δ subset of APER . On the other hand, if \mathcal{R} is hyperfinite then either $\text{APER} = \text{PARA}$ or $\text{APER} = \text{LOXO}$.*

This is a surprising result; by contrast consider the isometry group $\text{Isom}(\mathbb{H}^n)$ of real hyperbolic n -space. It is easy to show that the set of parabolic elements has positive codimension. Moreover, if Γ is a countable Gromov hyperbolic group then Γ does not have any parabolic elements at all.

1.3 Other results

1.3.1 Svarc-Milner Lemma

The Svarc-Milner Lemma is a fundamental result in geometric group theory. It states that if Γ is a finitely generated group acting isometrically and properly discontinuously on a metric space X and $x \in X$ then the map $\gamma \in \Gamma \mapsto \gamma x \in X$ is a quasi-isometric embedding with respect to any word metric on Γ . In particular, if X is a Gromov hyperbolic space and Γ acts isometrically and properly discontinuously with compact quotient then Γ itself is Gromov hyperbolic. In Theorem 2.5 we obtain the following analog: if the Main Assumption is satisfied and the section $\sigma : X \rightarrow \mathcal{H}$ is r -cobounded (meaning: the open radius r neighborhood of the orbit O_x^σ in \mathcal{H}_x is all of \mathcal{H}_x), then \mathcal{R} admits a hyperbolic graphing.

1.3.2 Minimality

We prove that if the Main Assumption is satisfied and \mathcal{R} is not hyperfinite then the action of \mathcal{R} on the bundle of limit sets \mathcal{L} is minimal in two different senses. First, suppose $K \subset \mathcal{L}$ is a Borel subset such that for a.e. $x \in X$, $K_x = K \cap \mathcal{L}_x$ is closed in \mathcal{L}_x . Suppose also that K is invariant in the sense that $\alpha(x, y)K_y = K_x$ for $(x, y) \in \mathcal{R}$. Then either K_x is empty for a.e. x or $K_x = \mathcal{L}_x$ for a.e. x . This is Theorem 2.9. It plays a key role in proving the existence of loxodromic elements of $[\mathcal{R}]$. Second, as mentioned above, the full group $[\mathcal{R}]$ acts minimally on $\text{Prob}(\mathcal{L} \rightarrow X)$ the space of fields of boundary measures. This is Corollary 7.3.

1.3.3 Limit sets

If G is a rank 1 simple Lie group then $g \in G$ is either elliptic, parabolic or loxodromic depending on whether it has 0, 1 or 2 fixed points on the boundary. These fixed points form the limit set of the subgroup $\langle g \rangle$. This observation leads one to guess that if $\mathcal{S} \leq \mathcal{R}$ is ergodic and hyperfinite then it should be parabolic or loxodromic depending on whether for the limit set $\mathcal{L}_x(\mathcal{S})$ has one or two elements for a.e. x . We prove in Lemma 3.4 below that indeed, if the Main Assumption is satisfied and if \mathcal{S} is loxodromic then $|\mathcal{L}_x(\mathcal{S})| = 2$ for a.e. x . The question remains open if \mathcal{S} is parabolic. By contrast, if the measure μ on X is only required to be \mathcal{R} -quasi-invariant instead of \mathcal{R} -invariant then there are counterexamples (this was observed earlier by Kaimanovich [Kai04]).

1.3.4 Maximal hyperfinite subequivalence relations

The result mentioned above is used to prove: if the Main Assumption is satisfied and \mathcal{R} is parabolic (in particular it is hyperfinite) then all of its aperiodic subequivalence relations are also parabolic. In turn this result is used to prove Theorem 1.3 that every aperiodic hyperfinite subequivalence relation $\mathcal{S} \leq \mathcal{R}$ is contained in a *unique* maximal hyperfinite subequivalence relation \mathcal{M} . Indeed, we obtain an explicit description of \mathcal{M} as the stabilizer for a canonical \mathcal{S} -invariant field of boundary measures.

1.3.5 Rank 1

Let us say that a measured equivalence relation \mathcal{R} has **rank 1** if every aperiodic hyperfinite subequivalence relation $\mathcal{S} \leq \mathcal{R}$ is contained in a *unique* maximal hyperfinite subequivalence relation. This definition is motivated by the theory of semisimple Lie groups: a semisimple Lie group G of noncompact type has real rank 1 if and only if every closed noncompact unimodular amenable subgroup is contained in a unique maximal unimodular amenable subgroup.

In Theorem 4.4 we prove that if \mathcal{R} has rank 1 and $\mathcal{S} \leq \mathcal{R}$ is an aperiodic hyperfinite subequivalence relation then the subequivalence relation generated by \mathcal{S} and the normalizer of $[\mathcal{S}]$ in $[\mathcal{R}]$ is hyperfinite. (In fact, we prove a more general result using quasi-normalizers).

Also we provide examples of equivalence relations that are not rank 1: let $G = SL(n, \mathbb{Z})$ ($n \geq 3$) or $G = H_1 \times H_2$ where H_1 contains an infinite amenable subgroup and H_2 is non-amenable. Let \mathcal{R} be the orbit-equivalence of an essentially free ergodic pmp G -action. In Corollaries 4.6, 4.7 we show that \mathcal{R} does not have rank 1. It follows that \mathcal{R} does not admit a hyperbolic graphing (as in Example 1). The latter fact is a well-known result of Adams [Ada94] that has been partially generalized and reproven in [Hjo08, PP00, Gab00] (for example, [Gab00] shows non-amenable groups with a cost 1 action are not treeable).

1.4 Related literature

The paper owes a large debt to Kaimanovich's paper [Kai04]. The latter studies amenable equivalence relations that admit hyperbolic graphings (as in Example 1). However, the measure is only assumed to be quasi-invariant under the relation rather than invariant. Kaimanovich proves that any such equivalence relation necessarily has an invariant field of boundary measures ν such that the support of ν_x has cardinality at most 2. The proof strategy has roots in earlier work of Adams [Ada94] which shows that an orbit-equivalence relation of a pmp essentially free action of a non-elementary hyperbolic group cannot decompose as a nontrivial direct product of measured equivalence relations. Corollary 4.7 below partially generalizes this result. There is also a related result of Adams showing that any cocycle of a higher rank semisimple Lie group into the isometry group of a hyperbolic space must be cohomologous to a cocycle taking values in a compact subgroup [Ada96].

A recent paper [AH11] of Anderegge and Henry studies isometric actions of equivalence relations on bundles of CAT(0) spaces. Unfortunately, I only learned about their paper when this paper was nearly finished. In addition to their results on isometric CAT(0) actions, it develops the general theory of fields of metric spaces over a Borel space much more thoroughly than is done in this paper.

As mentioned above, this paper proves that if \mathcal{R} satisfies the Main Assumption and $\mathcal{S} \leq \mathcal{R}$ is hyperfinite and aperiodic then the subequivalence relation \mathcal{S}' generated by \mathcal{S} and the normalizer of $[\mathcal{S}]$ in $[\mathcal{R}]$ is also hyperfinite. There is a well-known analogous result in the theory of von Neumann algebras due to Ozawa [Oza04]: If Γ is an i.c.c. (infinite conjugacy classes) Gromov hyperbolic group, then its von Neumann algebra, denoted $L\Gamma$ is solid, i.e., $A' \cap L\Gamma$ is amenable for every diffuse von Neumann subalgebra $A \subset L\Gamma$. This result was strengthened in [CS13].

1.5 Organization

Standard definitions and results on Gromov hyperbolic spaces and measured equivalence relations are relegated to Appendices A and B for easy reference.

Theorem 1.3 (that every aperiodic hyperfinite subequivalence relation is contained in a *unique* maximal hyperfinite subequivalence relation) is proven in §4. The proof uses all of the material in §2 - §4 except for the Svarc-Milner Lemma (Theorem 2.5), the Minimality Theorem 2.9 and Theorem 3.1 on the topology of $\text{Prob}(\mathcal{L} \rightarrow X)$ (the space of fields of boundary measures).

The proof of the Tit's Alternative (Theorem 1.2) is contained in §6. The proof relies on the existence of loxodromic elements (Theorem 5.1) which relies on the Minimality Theorem 2.9.

The existence of parabolic elements (Lemma 7.2) relies on the Tit's Alternative and a result of [CMTD14] on the existence of one-ended forests. The abundance of parabolic elements (Theorem 7.1) uses the topological structure of $\text{Prob}(\mathcal{L} \rightarrow X)$, proofs of which are in Appendix C. The final result, that the full group $[\mathcal{R}]$ acts minimally on $\text{Prob}(\mathcal{L} \rightarrow X)$ uses the existence of parabolic elements and everything in Appendix C.

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2 Bundles of hyperbolic spaces

Let (X, μ, \mathcal{R}) be a discrete ergodic pmp equivalence relation, (\mathcal{H}, d, π) a separable bundle of hyperbolic spaces over X and α an isometric action of \mathcal{R} on (\mathcal{H}, d, π) .

2.1 The boundary extension

In this section we define a Borel structure on $\overline{\mathcal{H}}$, the bundle of Gromov completions, and extend the action α to this bundle. Recall that each fiber \mathcal{H}_x may be regarded as a Gromov hyperbolic space with respect to the metric d_x which denotes the restriction of d to $\mathcal{H}_x \times \mathcal{H}_x$. Thus it has a Gromov boundary, denoted $\partial\mathcal{H}_x$ and a Gromov completion, denoted $\overline{\mathcal{H}}_x := \mathcal{H}_x \cup \partial\mathcal{H}_x$. We let $\overline{\mathcal{H}} = \sqcup_x \overline{\mathcal{H}}_x$, $\partial\mathcal{H} = \sqcup_x \partial\mathcal{H}_x \subset \overline{\mathcal{H}}$ denote the disjoint unions. We extend the projection map π to $\overline{\mathcal{H}}$ in the obvious way: $\pi(\xi) = x$ if $\xi \in \overline{\mathcal{H}}_x$.

Recall from Appendix A that the **Gromov product** of $p, q \in \mathcal{H}_x$ with respect to $r \in \mathcal{H}_x$ is defined by

$$(p|q)_r = (1/2)(d(p, r) + d(q, r) - d(p, q)).$$

It extends to the boundary by taking limits inferior (see Appendix A). Now we choose the Borel structure on $\overline{\mathcal{H}}$ to be the smallest one such that for any Borel sections $\sigma, \eta : X \rightarrow \mathcal{H}$ and Borel set $B \subset [0, \infty)$ the subset

$$\Omega(\sigma, \eta, B) := \{\xi \in \overline{\mathcal{H}} : \pi(\xi) = x \Rightarrow (\xi|\sigma(x))_{\eta(x)} \in B\}$$

is Borel.

Lemma 2.1. *A section $\xi : X \rightarrow \overline{\mathcal{H}}$ is Borel if and only if for every pair of Borel sections $\sigma, \eta : X \rightarrow \mathcal{H}$ the map $F : X \rightarrow [0, \infty)$ defined by*

$$F(x) = (\xi(x)|\sigma(x))_{\eta(x)}$$

is Borel.

Proof. Let $\sigma, \eta : X \rightarrow \mathcal{H}$ be Borel sections and $B \subset [0, \infty)$ be Borel. For any section $\xi : X \rightarrow \overline{\mathcal{H}}$,

$$F^{-1}(B) = \xi^{-1}(\Omega(\sigma, \eta, B)).$$

If ξ is Borel then $F^{-1}(B)$ is Borel which implies (because B is arbitrary) that F is Borel. On the other hand, if F is Borel then $\xi^{-1}(\Omega(\sigma, \eta, B))$ is Borel which implies that ξ is Borel (because the sets $\Omega(\sigma, \eta, B)$ generate the Borel sigma-algebra on $\overline{\mathcal{H}}$). \square

Lemma 2.2. *If $\xi, \sigma : X \rightarrow \overline{\mathcal{H}}$ and $\eta : X \rightarrow \mathcal{H}$ are Borel sections then the map*

$$x \mapsto (\xi(x)|\sigma(x))_{\eta(x)}$$

is Borel.

Proof. If σ maps into \mathcal{H} then this follows from the previous lemma. So without loss of generality, we may assume σ maps into $\partial\mathcal{H}$. Because $\mathcal{H} \rightarrow X$ is a separable bundle, there exists a sequence $\{\sigma_n\}_{n=1}^\infty$ of Borel sections such that for every $x \in X$, $\{\sigma_n(x)\}_{n \in \mathbb{N}}$ is dense in \mathcal{H}_x .

For $n \in \mathbb{N}$ and $x \in X$, let

$$K_x(n) = \{p \in \mathcal{H}_x : (p|\sigma(x))_{\eta(x)} \geq n\}.$$

Then $K(n) = \cup_x K_x(n)$ is Borel. Intuitively, $K_x(n)$ is a small neighborhood of $\sigma(x)$ when n is large. Let $f_n : X \rightarrow \mathbb{R}$ denote the function

$$f_n(x) = \inf(\xi(x)|\sigma_m(x))_{\eta(x)}$$

where the infimum is over all m such that $\sigma_m(x) \in K_x(n)$. Because $K(n)$ is Borel, f_n is also Borel. Because

$$(\xi(x)|\sigma(x))_{\eta(x)} = \liminf_{n \rightarrow \infty} f_n(x)$$

the lemma follows. \square

Lemma 2.3. *There exists a constant $\delta > 0$ such that \mathcal{H}_x is δ -hyperbolic for a.e. x .*

Proof. Because \mathcal{H} is separable there exists a sequence $\{\sigma_i\}_{i \in \mathbb{N}}$ of Borel sections $\sigma_i : X \rightarrow \mathcal{H}$ such that for every x , $\{\sigma_i(x)\}_{i \in \mathbb{N}}$ is dense in \mathcal{H}_x . Let

$$\delta_x := \sup_{i,j,k,l} \min\{(\sigma_i(x)|\sigma_j(x))_{\sigma_l(x)}, (\sigma_j(x)|\sigma_k(x))_{\sigma_l(x)}\} - (\sigma_i(x)|\sigma_k(x))_{\sigma_l(x)}.$$

This formula shows $x \mapsto \delta_x$ is Borel. Because $\{\sigma_i(x)\}_{i \in \mathbb{N}}$ is dense in \mathcal{H}_x , it follows that δ_x is a hyperbolicity constant for \mathcal{H}_x . Because $x \mapsto \delta_x$ is \mathcal{R} -invariant (meaning $\delta_x = \delta_y$ whenever $x\mathcal{R}y$), μ -ergodicity of \mathcal{R} implies the lemma. \square

Now let $\epsilon > 0$ be such that $\epsilon\delta \leq 1/5$ where $\delta > 0$ is a hyperbolicity constant for the fibers \mathcal{H}_x (for a.e. x). Given a Borel section $\sigma : X \rightarrow \mathcal{H}$, we define the metric $\rho_\sigma : \overline{\mathcal{H}} * \overline{\mathcal{H}} \rightarrow [0, \infty)$ by

$$\rho_\sigma(\xi, \eta) = \begin{cases} \inf \sum_{i=1}^{n-1} \exp(-\epsilon(\xi_i | \xi_{i+1})_{\sigma(x)}) & \xi \neq \eta \\ 0 & \xi = \eta \end{cases}$$

where the infimum is over all sequences $\xi_1, \dots, \xi_n \in \overline{\mathcal{H}}_x$ with $\xi_1 = \xi, \xi_n = \eta$. By Lemma A.1, the restriction of ρ_σ to any fiber $\overline{\mathcal{H}}_x$ is a metric. The previous lemmas imply that ρ_σ is Borel.

We extend the action α to $\overline{\mathcal{H}}$ by continuity. More precisely, because each $\alpha(x, y) : \mathcal{H}_y \rightarrow \mathcal{H}_x$ is an isometry there is a unique extension (which we also denote by $\alpha(x, y)$) from $\overline{\mathcal{H}}_y \rightarrow \overline{\mathcal{H}}_x$ that is a homeomorphism. The uniqueness of the extension implies that the cocycle equation $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$ is still satisfied.

Lemma 2.4.

$$\{(p, q) \in \overline{\mathcal{H}} \times \overline{\mathcal{H}} : \alpha(\pi(q), \pi(p))p = q\}$$

is a Borel subset of $\overline{\mathcal{H}} \times \overline{\mathcal{H}}$.

Proof. Let $\sigma : X \rightarrow \mathcal{H}$ be a Borel section and define the metric ρ_σ as above. Let D denote the set of all $(p, q) \in \mathcal{H} \times \mathcal{H}$ such that $\alpha(\pi(q), \pi(p))p = q$. This set is Borel by hypothesis on α . For $n > 0$, let D_n denote a certain $1/n$ -neighborhood of D in the square of the Gromov completions:

$$D_n := \{(p, q) \in \overline{\mathcal{H}} \times \overline{\mathcal{H}} : \exists(p', q') \in D \rho_\sigma(p, p') < 1/n, \rho_\sigma(q, q') < 1/n\}.$$

Because ρ_σ is Borel, D_n is Borel. The lemma now follows from:

$$\bigcap_n D_n = \{(p, q) \in \overline{\mathcal{H}} \times \overline{\mathcal{H}} : \alpha(\pi(q), \pi(p))p = q\}.$$

\square

2.2 Svarc-Milner

Here we prove an analogue of the Svarc-Milner Lemma, giving conditions under which a hyperbolic graphing exists.

Theorem 2.5. *Suppose the Main Assumption is satisfied and there is a metrically proper Borel section $\sigma : X \rightarrow \mathcal{H}$ and a real number $r > 0$ such that σ is r -cobounded: the open radius r neighborhood of the orbit $\mathcal{O}_x^\sigma = \{\alpha(x, y)\sigma(y) : y \in [x]_{\mathcal{R}}\}$ in \mathcal{H}_x is all of \mathcal{H}_x . Then \mathcal{R} admits a hyperbolic graphing.*

Remark 4. We do not actually need all of the Main Assumption. We do not use separability or compactness of the orbit closure.

Proof. Recall the definition of d_σ from Definition 4. Let $\mathcal{G} = \{(x, y) \in \mathcal{R} : d_\sigma(x, y) \leq 3r\}$. We claim that \mathcal{G} is a hyperbolic graphing. To see that \mathcal{G} is a graphing, let $(x, y) \in \mathcal{R}$. Let γ be a geodesic in \mathcal{H}_x from $\sigma(x)$ to $\alpha(x, y)\sigma(y)$. So γ is a continuous map from the interval $[0, d_\sigma(x, y)]$ to \mathcal{H}_x with $\gamma(0) = \sigma(x)$ and $\gamma(d_\sigma(x, y)) = \alpha(x, y)\sigma(y)$. Because σ is r -cobounded, for every number $t \in [0, d_\sigma(x, y)]$ there exists $y_t \in [x]_{\mathcal{R}}$ such that

$$d(\gamma(t), \sigma_x(y_t)) \leq r.$$

By the triangle inequality, if $t, s \in [0, d_\sigma(x, y)]$ then

$$d_\sigma(y_t, y_s) \leq 2r + |t - s|.$$

Therefore, if $|t - s| \leq r$ then $(y_t, y_s) \in \mathcal{G}$. In particular, $x, y_r, y_{2r}, \dots, y_{nr}, y$ is a path in \mathcal{G} from x to y where $n = \lfloor d_\sigma(x, y)/r \rfloor$. This proves \mathcal{G} is a graphing. Moreover if $d_{\mathcal{G}}(x, y)$ is the length of the shortest path in \mathcal{G} from x to y then

$$d_{\mathcal{G}}(x, y) \leq 2 + d_\sigma(x, y)/r.$$

On the other hand, there exists a shortest path $x = x_1, \dots, x_m = y$ in \mathcal{G} from x to y . By definition $d_\sigma(x_i, x_{i+1}) \leq 3r$. Therefore $d_\sigma(x, y) \leq 3rm = 3rd_{\mathcal{G}}(x, y)$. So we have proven

$$(1/3r)d_\sigma(x, y) \leq d_{\mathcal{G}}(x, y) \leq 2 + d_\sigma(x, y)/r.$$

This proves that $\sigma_x : [x]_{\mathcal{R}} \rightarrow \mathcal{H}_x$ is a quasi-isometry with respect to $d_{\mathcal{G}}$ and $d \upharpoonright \mathcal{H}_x$. Because σ is metrically proper, \mathcal{G} is locally finite. By [Väi05, Theorem 3.18], $([x]_{\mathcal{R}}, d_{\mathcal{G}})$ is a hyperbolic metric space. \square

2.3 Orbits & limit sets

Let $\sigma : X \rightarrow \mathcal{H}$ be a Borel section and define $\sigma_x, \mathcal{O}_x^\sigma, \mathcal{L}_x^\sigma$ as in the introduction. The main result of this section is:

Theorem 1.1 *If the Main Assumption is satisfied and $\sigma, \eta : X \rightarrow \mathcal{H}$ are metrically proper sections whose orbit closures $\overline{\mathcal{O}_x^\sigma}, \overline{\mathcal{O}_x^\eta}$ are compact (for a.e. x) then $\mathcal{L}_x^\sigma = \mathcal{L}_x^\eta$ for a.e. x .*

The idea of the proof is this: suppose $K \subset \overline{\mathcal{H}}$ is such that $K_x := K \cap \overline{\mathcal{H}}_x$ is closed and quasi-convex and K is invariant in the sense that $\alpha(x, y)K_y = K_x$. Suppose as well that $K_x \cap \mathcal{H}_x$ is nonempty for a.e. x . We consider the projection map from $\overline{\mathcal{H}}_x$ to K_x . In case K_x does not contain the limit set \mathcal{L}_x^σ , this map is infinite-to-1 (a.e. x). However, this contradicts the Mass-Transport Principle. Applying this to the case when K equals the convex hull of \mathcal{O}^σ , we obtain that $\mathcal{L}_x^\eta \subset \text{Hull}(\mathcal{O}_x^\sigma) \cap \partial \mathcal{H}_x$ for a.e. x . Since $\text{Hull}(\mathcal{O}_x^\sigma) \cap \partial \mathcal{H}_x = \mathcal{L}_x^\sigma$, this proves one inclusion. The other follows by symmetry.

Now the details. First we review convex hulls and quasi-convexity.

Definition 6. Let (\mathcal{H}, d) be a complete δ -hyperbolic metric space and $\overline{\mathcal{H}}$ its Gromov completion. A subset $Y \subset \overline{\mathcal{H}}$ is **C -quasi-convex** if for every $x, y \in Y$, every geodesic $[x, y]$ is contained in the radius- C -neighborhood of Y . If Y is C -quasi-convex for some $C > 0$ then Y is called **quasi-convex**. If $Y \subset \overline{\mathcal{H}}$ then the **closed convex hull** of Y , denoted $\text{Hull}(Y)$, is the closure of the union of all geodesic segments $[\xi, \eta]$ with $\xi, \eta \in Y$.

Lemma 2.6. *Let (\mathcal{H}, d) be a complete δ -hyperbolic metric space and $K \subset \overline{\mathcal{H}}$ be compact. Then $\text{Hull}(K)$ is quasi-convex and closed. Moreover, $\text{Hull}(K) \cap \partial \mathcal{H} = K \cap \partial \mathcal{H}$.*

Proof. The special case in which $K \subset \partial \mathcal{H}$ is [KS96, Lemmas 3.2, 3.6]. The general case is nearly identical. \square

Lemma 2.7. *Let (\mathcal{H}, d) be a complete δ -hyperbolic metric space and let $K \subset \overline{\mathcal{H}}$ be closed and C -quasi-convex. For every $x \in \mathcal{H}$, let $\text{Proj}_K(x)$ be the set of all elements $y \in K$ realizing the minimum distance from x to K (so $d(x, y) = d(x, K)$). For $\xi \in \partial \mathcal{H} \setminus K$ let $\text{Proj}_K(\xi)$ be the set of all $k \in K$ such that there exists a sequence $\{x_n\} \subset \mathcal{H}$ with $\lim_n x_n = \xi$ and $k_n \in \text{Proj}_K(x_n)$ with $\lim_n k_n = k$.*

Then there exists a constant $Q = Q(C, \mathcal{H}, d)$ such that for every $x \in \overline{\mathcal{H}} \setminus K$, the diameter of $\text{Proj}_K(x)$ is at most Q . Moreover, $\text{Proj}_K(x)$ is nonempty and if $\{x_n\} \subset \mathcal{H}$ is any sequence

with $\lim_n x_n = \xi \in \partial\mathcal{H} \setminus K$ then

$$\limsup_n d(\text{Proj}_K(x_n), \text{Proj}_K(\xi)) \leq Q$$

where $d(\cdot, \cdot)$ is the Hausdorff metric.

Proof. If $x \in \mathcal{H}$ then the fact that $\text{diam}(\text{Proj}_K(x))$ is uniformly bounded is [BH99, Chapter III.Γ Proposition 3.11]. The case $x \in \partial\mathcal{H} \setminus \partial K$ follows by taking limits. The rest of the lemma follows from standard arguments left to the reader. \square

The next lemma is used several times throughout the paper.

Lemma 2.8. *Suppose σ is metrically proper. Suppose $K \subset \overline{\mathcal{H}}$ is Borel and*

- $K_x := K \cap \overline{\mathcal{H}}_x$ is closed and quasi-convex for a.e. x ,
- $K_x \cap \mathcal{H}_x$ is nonempty for a.e. x ,
- $\alpha(y, x)K_x = K_y$ for all $x\mathcal{R}y$.

Then $\mathcal{L}_x^\sigma \subset K_x$ for a.e. x .

Proof. As in Lemma 2.7, given $b \in \mathcal{H}_x$, let

$$\text{Proj}_{K_x}(b) = \left\{ c \in K_x : d(c, b) = \inf\{d(c', b) : c' \in K_x\} \right\}.$$

We also extend Proj_{K_x} to $\partial\mathcal{H}_x$ by defining $\text{Proj}_{K_x}(\xi)$ equal to the set of all $h \in K_x$ such that there exists $\{q_n\} \subset \mathcal{H}_x$ with $q_n \rightarrow \xi$ and $h_n \in \text{Proj}_{K_x}(q_n)$ with $h_n \rightarrow h$ as $n \rightarrow \infty$.

By Lemma 2.7 there is a constant $Q > 0$ such that $\text{diam}(\text{Proj}_{K_x}(b)) \leq Q$ and $\text{Proj}_{K_x}(b)$ is nonempty for $b \in \overline{\mathcal{H}}_x \setminus K_x$.

Let W be the set of all $x \in X$ such that there exists $\xi \in \mathcal{L}_x^\sigma \setminus K_x$. To obtain a contradiction, suppose $\mu(W) > 0$. For $r > 0$, consider the subset

$$V_r := \{x \in X : \exists \xi \in \mathcal{L}_x^\sigma \setminus K_x \text{ such that } d(\sigma(x), \text{Proj}_{K_x}(\xi)) < r\}.$$

Because $\cup_r V_r = W$, there exists some $r_1 > 0$ such that $\mu(V_{r_1}) > 0$.

For $S \subset \mathcal{H}_x$ and $r > 0$ let $\mathcal{N}_r(S)$ denote the closed radius r -neighborhood of S . Define $F : \mathcal{R} \rightarrow \mathbb{R}$ by

$$F(x, y) = |\mathcal{O}_y^\sigma \cap \mathcal{N}_{r_1+Q}(\text{Proj}_{K_y}(\sigma_y(x)))|^{-1}$$

if $\sigma(y) \in \mathcal{N}_{r_1+Q}(\text{Proj}_{K_y}(\sigma_y(x)))$ and 0 otherwise. By the Mass-Transport Principle (Lemma B.1),

$$1 = \int \sum_{y \in [x]_{\mathcal{R}}} F(x, y) d\mu(x) = \int \sum_{x \in [y]_{\mathcal{R}}} F(x, y) d\mu(y). \quad (1)$$

Here we have used that σ is metrically proper and $\text{Proj}_{K_x}(\sigma(x))$ has bounded diameter to conclude that $|\mathcal{O}_y^\sigma \cap \mathcal{N}_{r_1+Q}(\text{Proj}_{K_y}(\sigma_y(x)))|$ is finite.

Let $y \in V_{r_1}$. By definition, there exists $\xi \in \mathcal{L}_y^\sigma \setminus K_y$ such that $d(\sigma(y), \text{Proj}_{K_y}(\xi)) < r_1$. Since ξ is a limit point, there is a sequence $\{x_i\} \subset [y]_{\mathcal{R}}$ such that $\sigma_y(x_i) \rightarrow \xi$ as $i \rightarrow \infty$.

By Lemma 2.7,

$$\limsup_i d(\text{Proj}_{K_y}(\sigma_y(x_i)), \text{Proj}_{K_y}(\xi)) \leq Q.$$

Therefore, if i is sufficiently large then $F(x_i, y) > 0$. By definition we have

$$F(x_i, y) = |\mathcal{O}_y^\sigma \cap \mathcal{N}_{r_1+Q}(\text{Proj}_{K_y}(\sigma_y(x_i)))|^{-1} \geq |\mathcal{O}_y^\sigma \cap \mathcal{N}_{r_1+3Q}(\text{Proj}_{K_y}(\xi))|^{-1} > 0.$$

Therefore

$$\sum_{x \in [y]_{\mathcal{R}}} F(x, y) \geq \sum_{x \in [y]_{\mathcal{R}}} |\mathcal{O}_y^\sigma \cap \mathcal{N}_{r_1+3Q}(\text{Proj}_{K_y}(\xi))|^{-1} = +\infty.$$

By (1), we must have that $\mu(V_{r_1}) = 0$ contradicting our earlier hypothesis. Thus $\mu(W) = 0$. This implies the lemma. □

Proof of Theorem 1.1. By Lemmas 2.6 and 2.8, for a.e. x ,

$$\mathcal{L}_x^\sigma \subset \text{Hull}(\mathcal{O}_x^\eta) \cap \partial \mathcal{H}_x = \mathcal{L}_x^\eta.$$

By symmetry, this proves the Theorem. □

From now on, we will write \mathcal{L} to mean \mathcal{L}^σ .

2.4 Minimality

Theorem 2.9. *Suppose the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite. Then the action of \mathcal{R} on the limit set \mathcal{L} is minimal in the following sense: if $C \subset \partial \mathcal{L}$ is Borel and*

- $C_x := C \cap \mathcal{L}_x^\sigma$ is closed for a.e. x ,

- C is α -invariant in the sense that $\alpha(x, y)C_y = C_x$ for a.e. $(x, y) \in \mathcal{R}$,
- $C_x \neq \emptyset$ for a.e. x

then $\mathcal{L}_x \subset C_x$ for a.e. x .

Proposition 2.10. *Suppose $\sigma : X \rightarrow \mathcal{H}$ is a metrically proper section and $\xi : X \rightarrow \partial \mathcal{H}$ is an \mathcal{R} -invariant section (so $\alpha(y, x)\xi(x) = \xi(y)$ a.e.). Then \mathcal{R} is hyperfinite.*

Proof. The proof is essentially the same as the proof of [Kai04, Theorem 3.4] (which uses Reiter's criterion to prove \mathcal{R} is amenable). The main difference is that Kaimanovich considers only the case when the bundle \mathcal{H} comes from a hyperbolic graphing as in Example 1. The general case is not significantly different. \square

Proof of Theorem 2.9. By Proposition 2.10 and the μ -ergodicity of \mathcal{R} , it follows that $|C_x| > 1$ for a.e. x . Therefore, $\text{Hull}(C_x) \cap \mathcal{H}_x$ is nonempty for a.e. x . So Lemma 2.8 implies $\mathcal{L}_x \subset C_x$ for a.e. x which implies the theorem. \square

3 Fields of boundary measures

Let us assume the Main Assumption is satisfied. As in the previous section, we let $\mathcal{L} \rightarrow X$ denote the bundle of limit sets. Given a function $F : \mathcal{L} \rightarrow \mathbb{C}$ and $x \in X$, we let $F_x : \mathcal{L}_x \rightarrow \mathbb{C}$ denote its restriction to the fiber \mathcal{L}_x . We say two functions F, G on \mathcal{L} are **equivalent** if for a.e. $x \in X$ $F_x = G_x$. For each $x \in X$, let $C(\mathcal{L}_x)$ denote the Banach space of continuous functions on \mathcal{L}_x with the sup norm. Suppose $F : \mathcal{L} \rightarrow \mathbb{C}$ is a Borel function such that $F_x \in C(\mathcal{L}_x)$ for a.e. $x \in X$. Then we define its norm by

$$\|F\| := \|x \mapsto \|F_x\|\|_{L^\infty(X, \mu)}.$$

Let $C(\mathcal{L} \rightarrow X)$ denote the set of all equivalence classes of Borel functions $F : \mathcal{L} \rightarrow \mathbb{C}$ such that for a.e. $x \in X$, $F_x \in C(\mathcal{L}_x)$ and $\|F\| < \infty$. This is a Banach space with the above norm.

A **field of boundary measures** is a collection $\nu = \{\nu_x : x \in X\}$ such that

- for every $x \in X$, ν_x is a Borel probability measure on the fiber \mathcal{L}_x and

- for every $F \in C(\mathcal{L} \rightarrow X)$, the map $x \mapsto \nu_x(F_x) = \int F_x d\nu_x$ is μ -measurable.

Two fields ν, η are **equivalent** if $\nu_x = \eta_x$ for a.e. x . By abusing notation, we will not distinguish between equivalent fields.

Let $\text{Prob}(\mathcal{L} \rightarrow X)$ denote the set of all (equivalence classes of) fields of boundary measures. Given $F \in C(\mathcal{L} \rightarrow X)$ and an open set $O \subset \mathbb{C}$, let $\Omega(F, O)$ be the set of all $\nu \in \text{Prob}(\mathcal{L} \rightarrow X)$ such that $\int \nu_x(F_x) d\mu(x) \in O$. We give $\text{Prob}(\mathcal{L} \rightarrow X)$ the topology generated by sets of the form $\Omega(F, O)$.

Theorem 3.1. *If the Main Assumption is satisfied then $\text{Prob}(\mathcal{L} \rightarrow X)$ is affinely homeomorphic to a compact convex metrizable subspace of a Banach space. Moreover the full group $[\mathcal{R}]$ acts jointly continuously on $\text{Prob}(\mathcal{L} \rightarrow X)$ by $(\phi\nu)_{\phi x} = \alpha(\phi x, x)_* \nu_x$.*

Proof. This follows from Lemma C.2 and Theorem C.5. □

Definition 7 (Fixed point set). If $\mathcal{S} \leq \mathcal{R}$ is a subequivalence relation then we let $\text{Fix}(\mathcal{S})$ denote the set of all $\eta \in \text{Prob}(\mathcal{L} \rightarrow X)$ such that for a.e. $(x, y) \in \mathcal{S}$, $\alpha(y, x)_* \eta_x = \eta_y$. Also if $f \in [\mathcal{R}]$ then we let $\text{Fix}(f)$ denote the set of all $\eta \in \text{Prob}(\mathcal{L} \rightarrow X)$ such that for a.e. x , $\alpha(f(x), x)_* \eta_x = \eta_{f(x)}$. Observe that $\text{Fix}(\mathcal{S})$ and $\text{Fix}(f)$ are closed convex subsets of $\text{Prob}(\mathcal{L} \rightarrow X)$.

Theorem 3.2. *Suppose the Main Assumption is satisfied. Let $\mathcal{S} \leq \mathcal{R}$ be a subequivalence relation. Then $\text{Fix}(\mathcal{S})$ is nonempty if and only if \mathcal{S} is hyperfinite. Moreover, if \mathcal{S} is aperiodic and $\nu \in \text{Fix}(\mathcal{S})$ then for a.e. x , the support of ν_x has at most 2 points.*

Remark 5. The proof of Theorem 3.2 above is essentially the same as [Kai04, Theorem 2.20]. The main difference between the two results are in the hypotheses. In Kaimanovich's case the bundle \mathcal{H} comes from a hyperbolic graphing (as in Example 1), the equivalence relation \mathcal{R} need not be measure-preserving and ν is assumed to be \mathcal{R} -invariant instead of merely \mathcal{S} -invariant. This strategy has roots in earlier work of Adams [Ada94].

Here is a brief sketch: if \mathcal{S} is hyperfinite then the fixed point property of amenable equivalence relations (see [Zim84]) immediately gives that $\text{Fix}(\mathcal{S})$ is nonempty. If $\text{Fix}(\mathcal{S})$ is nonempty then one can prove \mathcal{S} is hyperfinite from Reiter's condition. Now suppose $\nu \in \text{Fix}(\mathcal{S})$. There is a map that associates to any Borel probability measure on $\partial \mathcal{H}_x$ whose

support contains more than 2 elements, a bounded subset of \mathcal{H}_x called its barycenter. Let r_x be the infimum of the distances between $\sigma_x(y)$ and this barycenter. Because σ is metrically proper, the subset of $[x]_{\mathcal{R}}$ that realizes this infimum is a finite set called the σ -barycenter. Note we are using the metrical properness of σ as a surrogate for the local finiteness that plays a similar role in Adams' arguments.

The map that associates to x the σ -barycenter of ν_x is Borel and \mathcal{R} -invariant. Therefore, if the support of ν_x contains more than 2 elements for a set of x 's of positive measure then by Lemma B.3, \mathcal{S} is finite on a set of positive measure. In particular, it cannot be aperiodic.

Recall that a hyperfinite subequivalence relation $\mathcal{S} \leq \mathcal{R}$ is **parabolic** if there is a unique element $\nu \in \text{Fix}(\mathcal{S})$ and ν_x is supported on a single point of \mathcal{L}_x for a.e. x . We say \mathcal{S} is **loxodromic** if there exists $\nu \in \text{Fix}(\mathcal{S})$ such that the support of ν_x contains two points of \mathcal{L}_x for a.e. x . We say \mathcal{S} is **mixed** if there is a nontrivial measurable partition $X = Y \sqcup Z$ such that $\mathcal{S} \upharpoonright Y$ is parabolic and $\mathcal{S} \upharpoonright Z$ is loxodromic. Moreover this partition is unique up to null sets. Uniqueness implies it is \mathcal{S} -invariant (modulo a measure zero set).

Lemma 3.3. *Suppose the Main Assumption is satisfied and $\mathcal{S} \leq \mathcal{R}$ is aperiodic and hyperfinite. Then \mathcal{S} is either parabolic, loxodromic or mixed.*

Proof. If $\nu \in \text{Fix}(\mathcal{S})$ then let $D(\nu)$ be the set of all $x \in X$ such that ν_x is a Dirac measure on \mathcal{L}_x (that is, the ν_x is supported on a single point of \mathcal{L}_x). Let $\beta = \inf\{\mu(D(\nu)) : \nu \in \text{Fix}(\mathcal{S})\}$. For every natural number n , let $\nu_n \in \text{Fix}(\mathcal{S})$ be a field of boundary measures such that $\mu(D(\nu_n)) \leq \beta + 1/n$. Finally consider $\sum_{n=1}^{\infty} 2^{-n} \nu_n := \nu_{\infty}$. Because $\text{Fix}(\mathcal{S})$ is convex and closed, $\nu_{\infty} \in \text{Fix}(\mathcal{S})$ and $\mu(D(\nu_{\infty})) = \beta$ by construction.

If \mathcal{S} is neither loxodromic nor parabolic then $0 < \beta < 1$. Observe that \mathcal{S} restricted to $D(\nu_{\infty})$ is parabolic and \mathcal{S} restricted to the complement $X \setminus D(\nu_{\infty})$ is loxodromic. The uniqueness of the set $D(\nu_{\infty})$ is immediate since if $\mathcal{S} \upharpoonright Y_i$ is parabolic (for $i = 1, 2$) then $\mathcal{S} \upharpoonright (Y_1 \cup Y_2)$ is also parabolic. \square

3.1 Limit sets of hyperfinite equivalence relations

It may seem obvious that if \mathcal{R} is parabolic then the limit set $\mathcal{L}_x(\mathcal{R})$ should contain exactly one point and if \mathcal{R} is loxodromic then $\mathcal{L}_x(\mathcal{R})$ contains exactly two points. We prove the

latter below (assuming the Main Assumption); the former remains open. This result is used in the proof that every aperiodic hyperfinite subequivalence relation is contained in a unique maximal hyperfinite subequivalence relation.

Before getting to the proof we observe that both statements are definitely false if we do not require \mathcal{R} to be probability-measure-preserving. To see this, let G denote a non-elementary word hyperbolic group and $S \subset G$ a finite symmetric generating set for G . If $G \curvearrowright (X, \mu)$ is any measure-class-preserving action of G and $\mathcal{R} \subset X \times X$ is the resulting orbit-equivalence relation then there is a canonical graphing \mathcal{G} of X : $\mathcal{G} = \{(x, sx) : s \in S, x \in X\}$. This graphing determines a canonical bundle of hyperbolic metric spaces over X as in Example 1 in which each fiber is isometric to the Cayley graph determined by (G, S) . Now let $\Gamma = \Gamma(G, S)$ denote this Cayley graph, $\partial\Gamma$ its Gromov boundary and ν a probability measure on $\partial\Gamma$ whose measure class is preserved under the canonical G -action. In [Kai04, Examples 2.2.4, 2.2.5] it is shown that the orbit-equivalence relation of the action $G \curvearrowright (\partial\Gamma, \nu)$ is parabolic and the orbit-equivalence relation of $G \curvearrowright (\partial\Gamma \times \partial\Gamma, \nu \times \nu)$ is loxodromic with respect to the canonical hyperbolic bundle described above. (Neither equivalence relation is probability-measure-preserving and therefore neither satisfies the Main Assumption). However, in both cases each fiber is isometric to the Cayley graph Γ on which G acts vertex-transitively. It follows that in both cases, the limit set \mathcal{L}_x can be identified with $\partial\Gamma$. It is therefore infinite whenever G is non-elementary.

Lemma 3.4. *Suppose the Main Assumption holds, \mathcal{R} is loxodromic and $\nu \in \text{Fix}(\mathcal{R})$ is an invariant field of boundary measures such that the support of ν_x has cardinality 2 (for a.e. x) then \mathcal{L}_x is the support of ν_x for a.e. x . In particular, $|\mathcal{L}_x| = 2$ for a.e. x .*

Proof. For $r > 0$, let X_r denote the set of all $x \in X$ such that there exists a geodesic $\gamma_x \subset \mathcal{H}_x$ with endpoints in the support of ν_x such that $d(\sigma(x), \gamma_x) \leq r$. Because $\cup_{r>0} X_r = X$, there exists $r > 0$ such that $\mu(X_r) > 0$.

Let $\mathcal{O}_x^r := \{\sigma_x(y) : y \in X_r \cap [x]_{\mathcal{R}}\}$ and $\mathcal{L}_x^r := \overline{\mathcal{O}_x^r} \cap \partial\mathcal{H}_x$ be the associated limit set. On the one hand, because each $y \in X_r$ has image $\sigma_x(y)$ r -close to a geodesic γ_x , it follows that \mathcal{L}_x^r is contained in the support of ν_x . On the other hand, Lemma 2.8 implies

$$\mathcal{L}_x \subset \text{Hull}(\overline{\mathcal{O}_x^r}) \cap \partial\mathcal{H}_x = \mathcal{L}_x^r \subset \text{support}(\nu_x).$$

Since the support of ν_x is contained in \mathcal{L}_x by definition of $\text{Fix}(\mathcal{R})$, the lemma follows. \square

3.2 Parabolic equivalence relations

The goal of this section is to prove when \mathcal{R} is parabolic then every aperiodic subequivalence relation of \mathcal{R} also parabolic. More precisely:

Theorem 3.5. *Suppose the Main Assumption is satisfied and \mathcal{R} is hyperfinite and parabolic. Then every aperiodic subequivalence relation $\mathcal{S} \leq \mathcal{R}$ is also parabolic.*

Here is a short proof sketch: to obtain a contradiction suppose there exists a loxodromic $\mathcal{S} \leq \mathcal{R}$. Because \mathcal{R} is parabolic there exists a unique \mathcal{R} -invariant section $\xi : X \rightarrow \mathcal{L}$. We show there is a function $\Phi : X \rightarrow X$ with $x\mathcal{R}\Phi(x)$ (for a.e. x), such that $\sigma_x(\Phi(x))$ is “closer” to the boundary point $\xi(x)$ than $\sigma(x)$ and the three points $\sigma(x), \sigma_x(\Phi(x)), \xi(x)$ all lie close to a geodesic. Moreover the broken geodesic path

$$[\sigma(x), \sigma_x(\Phi(x))] \cup [\sigma_x(\Phi^2(x)), \sigma_x(\Phi^3(x))] \cup \dots$$

forms a quasi-geodesic limiting on $\xi(x)$. The Mass Transport Principle implies that Φ restricted to a certain positive measure subset of X is a bijection (mod μ). But this contradicts the geometric fact that Φ is contracting.

Now for the details. Recall that if $Y \subset X$ then $\mathcal{R} \upharpoonright Y := \mathcal{R} \cap (Y \times Y)$ is an equivalence relation on Y called the **compression** (or **restriction**) of \mathcal{R} to Y (more details of this construction are discussed in §B.3). It will be convenient to have the following Lemma:

Lemma 3.6. *Suppose the Main Assumption is satisfied and \mathcal{R} is hyperfinite. Let $Y \subset X$ have positive μ -measure. Then $\mathcal{R} \upharpoonright Y$ is parabolic if and only if \mathcal{R} is parabolic. Similarly, $\mathcal{R} \upharpoonright Y$ is loxodromic if and only if \mathcal{R} is loxodromic.*

Proof. Because \mathcal{R} and $\mathcal{R} \upharpoonright Y$ are ergodic, they cannot be mixed. So it suffices to show that \mathcal{R} is loxodromic if and only if $\mathcal{R} \upharpoonright Y$ is loxodromic. If \mathcal{R} is loxodromic then there exists a \mathcal{R} -invariant field of boundary measure ν such that the support of ν_x has cardinality two for a.e. x . Then ν restricted to Y is $\mathcal{R} \upharpoonright Y$ -invariant. This proves $\mathcal{R} \upharpoonright Y$ is loxodromic.

On the other hand, suppose $\mathcal{R} \upharpoonright Y$ is loxodromic. Let $\nu = \{\nu_x\}_{x \in Y}$ be a $\mathcal{R} \upharpoonright Y$ -invariant field of boundary measures such that the support of ν_x has cardinality two for a.e. $x \in Y$. For any $x \in X$, define ν_x by $\nu_x = \alpha(x, y)_* \nu_y$ for $y \in Y \cap [x]_{\mathcal{R}}$. This is well-defined because ν

is $\mathcal{R} \upharpoonright Y$ -invariant and μ is \mathcal{R} -ergodic (so for a.e. $x \in X$, there exists $y \in Y \cap [x]_{\mathcal{R}}$). Clearly the extended ν is \mathcal{R} -invariant. This proves \mathcal{R} is loxodromic. \square

Quasi-geodesics are defined in §A.2. We will use the following lemma to show that certain broken geodesic paths are quasi-geodesics:

Lemma 3.7. *Let $(\mathcal{H}, d_{\mathcal{H}})$ be a geodesic δ -hyperbolic space. Then for every $s > 0$ there exists an $r > 0$ such that if $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{H}$ is any sequence with*

$$d_{\mathcal{H}}(x_i, x_{i+1}) \geq r, \quad (x_i | x_{i+2})_{x_{i+1}} \leq s \quad \forall i \in \mathbb{N}$$

then the piecewise geodesic $[x_1, x_2] \cup [x_2, x_3] \cup \dots$ obtained by concatenating successive geodesic segments together is a quasi-geodesic in \mathcal{H} . In particular, $\lim_{i \rightarrow \infty} x_i \in \partial \mathcal{H}$ exists.

Proof. This follows from standard arguments. For example, see [BH99, Chapter III.H, Theorem 1.13]. \square

Proof of Theorem 3.5. Because \mathcal{R} is parabolic there exists a unique \mathcal{R} -invariant section $\xi : X \rightarrow \mathcal{L}$. To obtain a contradiction, suppose $\mathcal{S} \leq \mathcal{R}$ is loxodromic on a subset of positive measure. After passing to this subset if necessary, we may assume that \mathcal{S} is loxodromic. Let $\nu \in \text{Fix}(\mathcal{S})$ be such that the support of ν_x contains 2 points for a.e. x .

Claim 1. For a.e. $x \in X$, $\cup_{y \in \mathcal{R}x} \alpha(x, y) \mathcal{L}_y(\mathcal{S})$ is an infinite subset of $\mathcal{L}_x(\mathcal{R})$.

Proof of Claim 1. Suppose not. By ergodicity there is a natural number k such that $|\cup_{y \in \mathcal{R}x} \alpha(x, y) \mathcal{L}_y(\mathcal{S})| = k$ for a.e. x . Let η_x be the probability measure uniformly supported on $\cup_{y \in \mathcal{R}x} \alpha(x, y) \mathcal{L}_y(\mathcal{S})$. Then η is \mathcal{R} -invariant. By Theorem 3.2 this implies $k \leq 2$. By Lemma 3.4, $|\mathcal{L}_y(\mathcal{S})| = 2$ for a.e. y . So $k = 2$. However, this implies \mathcal{R} is loxodromic, a contradiction. \square

For $s > 0$, let X_s be the set of all $x \in X$ such that there exists a geodesic $\gamma_x \subset \mathcal{H}_x$ whose endpoints are in the support of ν_x satisfying $d(\sigma(x), \gamma_x) \leq s$. Then $\cup_{s>0} X_s = X$; so there exists $s > 0$ such that $\mu(X_s) > 0$. By Lemma 3.6, we may assume without loss of generality that $X = X_s$.

Let $r > 0$ be large enough so that if z_1, z_2, \dots is any sequence of points in the δ -hyperbolic space \mathcal{H}_x satisfying (a) $d(z_i, z_{i+1}) \geq r$ for all i and (b) $(z_i | z_{i+2})_{z_{i+1}} \leq 4s + 22\delta$ for all i then

the path obtained by concatenating geodesic segments $[z_i, z_{i+1}]$ is a quasi-geodesic. Such a number $r > 0$ exists by Lemma 3.7. Moreover it does not depend on x because μ is \mathcal{R} -ergodic.

Let $\beta : \mathcal{R} \rightarrow \mathbb{R}$ be the Busemann function associated to ξ defined by

$$\beta(y, z) = \sup_{\{p_i\}} \limsup_{i \rightarrow \infty} d(\sigma(y), p_i) - d(\sigma_y(z), p_i)$$

where the sup is over all sequences $\{p_i\}_{i \in \mathbb{N}} \subset \mathcal{H}_y$ such that $\lim_{i \rightarrow \infty} p_i = \xi(x)$.

Claim 2. For a.e. $x \in X$ there exists $y \in X$ satisfying $y\mathcal{R}x$ and

$$\max(d_\sigma(x, y) - 4s - 20\delta, r) \leq \beta(x, y) \leq d_\sigma(x, y).$$

Proof of Claim 2. In general $|\beta(x, y)| \leq d_\sigma(x, y)$ holds because of the triangle inequality. So the last inequality is immediate.

We claim that $\xi(x) \in \mathcal{L}_x(\mathcal{S})$ for a.e. x . By Lemma 3.4 $\mathcal{L}_x(\mathcal{S})$ is the support of ν_x . Define a field of boundary measures λ by

$$\lambda_x = \frac{\nu_x + \delta_{\xi(x)}}{2}.$$

If $\xi(x)$ is not in the support of ν_x then the support of λ has three elements in contradiction to Theorem 3.2. So $\xi(x) \in \mathcal{L}_x(\mathcal{S})$.

By choice of s , there exists a geodesic γ_x with endpoints in the support of ν_x and $d(\sigma(x), \gamma_x) \leq s$. Because $\xi(x) \in \mathcal{L}_x(\mathcal{S})$ there exists $y \in X$ with $y\mathcal{S}x$ and $r \leq \beta(x, y)$. Let $x', y' \in \gamma_x$ be points closest to $\sigma(x), \sigma_x(y)$, respectively. So $d(\sigma(x), x') \leq s$. By choice of s there exists a geodesic γ'_x with endpoints in the support of ν_x such that $d(\sigma_x(y), \gamma'_x) \leq s$. By Lemma A.3, γ_x and γ'_x are 2δ -close. So $d(\sigma_x(y), y') \leq s + 2\delta$.

By Lemma A.5, since x', y' lie on the same geodesic with an endpoint in $\xi(x)$, $\beta(x', y') \geq d(x', y') - 2\delta$. By the quasi-cocycle inequality (Lemma A.6) and the triangle inequality,

$$|\beta(x, y) - \beta(x', y')| \leq |\beta(x, x')| + |\beta(y, y')| + 8\delta \leq d(\sigma(x), x') + d(y', \sigma_x(y)) \leq 2s + 10\delta.$$

So we have

$$\beta(x, y) \geq \beta(x', y') - 2s - 10\delta \geq d(x', y') - 2s - 16\delta \geq d_\sigma(x, y) - 4s - 20\delta.$$

□

We would like to define a map $\Phi : X \rightarrow X$ by: $\sigma_x(\Phi(x))$ is the closest point in the orbit \mathcal{O}_x^σ to $\sigma(x)$ that lies on the geodesic ray $[\sigma(x), \xi(x)]$. But of course, the orbit might not intersect any such geodesic ray in more than one point. So we use the Busemann function associated to $\xi(x)$ to define what it means for a point to be “closer” to $\xi(x)$. Fix an injective Borel map $\phi : X \rightarrow [0, 1]$ that we will use to break ‘ties’.

Now define $\Phi : X \rightarrow X$ by $\Phi(x) = y$ where $y \in [x]_{\mathcal{R}}$ satisfies

- $\max(d_\sigma(x, y) - 4s - 20\delta, r) \leq \beta(x, y)$,
- $d_\sigma(x, y) = \inf\{d_\sigma(x, z) : z \in [x]_{\mathcal{R}}, \max(d_\sigma(x, z) - 4s - 20\delta, r) \leq \beta(x, z)\}$
- if there is more than one element $y \in [x]$ satisfying the above conditions then we choose y to be the unique one minimizing $\phi(\cdot)$.

By Claim 2 and metrical properness of σ , Φ is well-defined.

To simplify notation, if x, y, z are any points in the same \mathcal{R} -class then we let

$$(x|y)_z^\sigma := (\sigma(x)|\sigma_x(y))_{\sigma_x(z)}.$$

Claim 3. For a.e. $x \in X$ the path obtained by concatenating geodesic segments $[\sigma_x(\Phi^n(x)), \sigma_x(\Phi^{n+1}(x))]$ ($n \in \mathbb{Z}$) is a quasi-geodesic and $\lim_{n \rightarrow \infty} \sigma_x(\Phi^n(x)) = \xi(x)$.

Proof of Claim 3. By the quasi-cocycle identity (Lemma A.6),

$$\begin{aligned} d_\sigma(\Phi^n x, \Phi^{n+2} x) &\geq \beta(\Phi^n x, \Phi^{n+2} x) \geq \beta(\Phi^n x, \Phi^{n+1} x) + \beta(\Phi^{n+1} x, \Phi^{n+2} x) - 4\delta \\ &\geq d_\sigma(\Phi^n x, \Phi^{n+1} x) + d_\sigma(\Phi^{n+1} x, \Phi^{n+2} x) - 8s - 44\delta. \end{aligned}$$

Thus

$$(\Phi^n(x)|\Phi^{n+2}(x))_{\Phi^{n+1}(x)}^\sigma \leq 4s + 22\delta.$$

Since $d_\sigma(\Phi^n(x), \Phi^{n+1}(x)) \geq \beta(\Phi^n(x), \Phi^{n+1}(x)) \geq r$, the Claim now follows from the choice of r . \square

For $x \in X$, let G_x be the directed graph with vertex set $[x]_{\mathcal{R}}$ and directed edges $(x, \Phi(x))$. Let G'_x be the induced subgraph of G_x containing all vertices that lie in a bi-infinite directed path of G_x . In other words, G'_x is the maximal subgraph of G_x that does not have any vertices of degree 1.

Claim 4. For a.e. $x \in X$, G'_x is nonempty. In fact, for every limit point $\eta \in \cup_{y \in \mathcal{R}_x} \alpha(x, y) \mathcal{L}_y(\mathcal{S})$ with $\eta \neq \xi(x)$, there exists a bi-infinite directed path in G'_x with endpoints $\{\xi(x), \eta\}$.

Proof of Claim 4. Let $y \in [x]_{\mathcal{R}}$. By Lemma 3.4 there is a point $\eta \neq \xi(x)$ such that $\alpha(x, y) \mathcal{L}_y(\mathcal{S}) = \{\eta, \xi(x)\}$ and elements $y_n \in [y]_{\mathcal{S}}$ such that $\lim_{n \rightarrow \infty} \sigma_x(y_n) = \eta$. Let H_n be the subgraph of G_x induced by the trajectory $\{\Phi^m(y_n)\}_{m \in \mathbb{N}}$. Observe that if $\gamma_x \subset \mathcal{H}_x$ is any geodesic from η to $\xi(x)$ then the distance between $\sigma_x(\Phi^m(y_n))$ and γ_x is bounded by a constant independent of n, m . This is because the path obtained by concatenating geodesic segments $[\sigma_x(\Phi^n(x)), \sigma_x(\Phi^{n+1}(x))]$ ($n \in \mathbb{Z}$) is a (λ, C) -quasi-geodesic limiting on $\xi(x)$ (for some $\lambda, C > 0$ independent of n) and $d(\sigma_x(y_n), \gamma_x) \leq s + 2\delta$. It follows that the subgraph H_n has a subsequential limit H_∞ in the space of all subgraphs of G_x (endowed with the topology of pointwise convergence on compact subsets) and H_∞ lies in the $(s + 2\delta)$ -neighborhood of γ_x . Since $y_n \rightarrow \eta$ as $n \rightarrow \infty$, this implies that H_∞ is the required bi-infinite directed path. \square

Let Z denote the set of all $x \in X$ such that x is a vertex of G'_x . Define $F : \mathcal{R} \rightarrow [0, 1]$ by $F(x, y) = 1$ if $x \in Z$ and $y = \Phi(x)$. Let $F(x, y) = 0$ otherwise. By the Mass-Transport Principle (Lemma B.1),

$$\mu(Z) \leq \int_Z \sum_x F(x, y) d\mu(y) \leq \int \sum_x F(x, y) d\mu(y) = \int \sum_y F(x, y) d\mu(x) = \mu(Z).$$

This implies $\sum_x F(x, y) = 1$ for a.e. $y \in Z$. So every vertex of G'_x is adjacent to exactly one incoming edge and one outgoing edge. In particular, every vertex of G'_x has degree 2. By Claims 1 and 4, this implies that G'_x has infinitely many connected components.

Claim 5. There is a constant $C > 0$ such that for any $y, z \in [x]_{\mathcal{R}}$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ there exists $m \in \mathbb{N}$ such that $d_\sigma(\Phi^n y, \Phi^m z) < C$.

Proof of Claim 5. Let γ_y be a geodesic from $\sigma_x(y)$ to $\xi(x)$. Because the path obtained by concatenating geodesic segments $[\sigma_x(\Phi^n(y)), \sigma_x(\Phi^{n+1}(y))]$ ($n \in \mathbb{Z}$) is a (λ, C) -quasi-geodesic limiting on $\xi(x)$ (for some $\lambda, C > 0$ independent of y), there is a constant C' such that $\{\sigma_x(\Phi^n(y))\}_{n \in \mathbb{N}}$ lies in the C' -neighborhood of γ_y . Similarly, if γ_z is a geodesic from $\sigma_x(z)$ to $\xi(x)$ then $\{\sigma_x(\Phi^n(z))\}_{n \in \mathbb{N}}$ lies in the C' -neighborhood of γ_z . The two geodesics γ_y and γ_z are asymptotic. By Lemma A.3 there is an N and constant $\delta' > 0$ such that for all $n > N$, $\sigma_x(\Phi^n(y))$ lies in the $(C' + 2\delta')$ -neighborhood of γ_z . The statement now follows from the definition of Φ . \square

For $n > 0$, let Y_n be the set of all $x \in X$ such that there exist at most n points $y \in [x]_{\mathcal{R}}$ with $d_{\sigma}(x, y) \leq C$ where $C > 0$ is as in Claim 5. Because σ is metrically proper, $X = \cup_{n=1}^{\infty} Y_n$. So there exists an n such that Y_n has positive measure. By replacing \mathcal{R} with $\mathcal{R} \upharpoonright Y_n$ and invoking Lemma 3.6 we may assume without loss of generality that $X = Y_n$.

We have now arrived at a contradiction. To be precise, let $x \in X$ be a typical element. By Claim 1, there exist distinct elements $\eta_1, \dots, \eta_{n+1} \in \cup_{y \in [x]_{\mathcal{R}}} \alpha(x, y) \mathcal{L}_y(\mathcal{S})$ such that $\eta_i \neq \xi(x)$ for all i . By Claim 4, for each i there is a directed path γ_i in G'_x with endpoints $\{\eta_i, \xi(x)\}$. By the paragraph before Claim 5, the paths $\gamma_1, \dots, \gamma_{n+1}$ are pairwise disjoint. By Claim 5, there exists $y \in [x]_{\mathcal{R}}$ and $y_i \in \gamma_i$ such that $d_{\sigma}(y, y_i) \leq C$ for all i . This contradicts the choice of n .

□

4 Maximal hyperfinite subequivalence relations

Definition 8. A hyperfinite subequivalence $\mathcal{S} \leq \mathcal{R}$ is **maximal hyperfinite** if there does not exist a hyperfinite subequivalence relation $\mathcal{S}' \leq \mathcal{R}$ with $\mathcal{S} \leq \mathcal{S}'$ and $\mathcal{S}' \setminus \mathcal{S}$ non-null (with respect to the usual measure $\hat{\mu}$ on \mathcal{R} , see §B).

We say \mathcal{R} has **rank 1** if every aperiodic hyperfinite subequivalence relation is contained in a *unique* maximal hyperfinite subequivalence relation. We will show that if the Main Assumption is satisfied then \mathcal{R} is rank 1 and characterize rank 1 in terms of quasi-normalizers of hyperfinite subequivalence relations. We also use this to prove that certain equivalence relations *do not* satisfy the Main Assumption.

Lemma 4.1. *If $\mathcal{S} \leq \mathcal{R}$ is any hyperfinite subequivalence relation then there exists a maximal hyperfinite subequivalence relation $\mathcal{S}' \leq \mathcal{R}$ with $\mathcal{S} \leq \mathcal{S}'$.*

Proof. We say that two subequivalence relations $\mathcal{S}_1, \mathcal{S}_2$ are **$\hat{\mu}$ -equivalent** if $\hat{\mu}(\mathcal{S}_1 \Delta \mathcal{S}_2) = 0$ where $\hat{\mu}$ is the usual measure on \mathcal{R} (see §B). In general, we will not distinguish between a subequivalence relation and its $\hat{\mu}$ -class. Let \mathcal{Z} denote the collection of all (equivalence classes of) Borel hyperfinite subequivalence relations \mathcal{S}' of \mathcal{R} with $\mathcal{S} \leq \mathcal{S}' \leq \mathcal{R}$. The set \mathcal{Z} is partially ordered by inclusion mod $\hat{\mu}$. By Zorn's Lemma there exists a maximal chain $\mathcal{C} \subset \mathcal{Z}$.

Let μ' be a probability measure on \mathcal{R} that is equivalent to $\widehat{\mu}$ (in the sense that two measures have the same measure zero sets). Let $\beta = \sup\{\mu'(\mathcal{T}) : \mathcal{T} \in \mathcal{C}\}$. For each integer $n \geq 1$, let \mathcal{S}_n be an element of the chain \mathcal{C} satisfying $\mu'(\mathcal{S}_n) \geq \beta - 1/n$. Since \mathcal{C} is a chain, we must have that $\mathcal{S}_1 \leq \mathcal{S}_2 \leq \dots$ (with inclusions taken mod $\widehat{\mu}$). Since each \mathcal{S}_i is hyperfinite, the union $\mathcal{S}_\infty := \cup_{i=1}^\infty \mathcal{S}_i$ is also hyperfinite. Moreover, $\mu'(\mathcal{S}_\infty) = \beta$. Since \mathcal{C} is a maximal chain, we must have that $\mathcal{S}_\infty \in \mathcal{C}$. Again since \mathcal{C} is a maximal chain and $\mu'(\mathcal{S}_\infty) = \beta$, \mathcal{S}_∞ must be a maximal hyperfinite subequivalence relation. \square

Definition 9. A pmp equivalence relation \mathcal{R} has **rank 1** if every pair of distinct maximal hyperfinite subequivalence relations intersects in a finite subequivalence relation. Equivalently, \mathcal{R} has rank 1 if every aperiodic hyperfinite subequivalence relation is contained in a unique maximal hyperfinite subequivalence relation (where uniqueness is up to measure zero). We say \mathcal{R} has **higher rank** if it does not have rank 1. This definition is motivated by the theory of semisimple Lie groups: a semisimple Lie group G has real rank 1 if and only if every closed noncompact unimodular amenable subgroup is contained in a unique maximal unimodular amenable subgroup.

Theorem 4.2. *Hyperbolic equivalence relations have rank 1.*

Definition 10. Every aperiodic hyperfinite subequivalence relation $\mathcal{S} \leq \mathcal{R}$ has a unique **canonical invariant field of boundary measures** defined as follows. By Lemma 3.3, there is a unique (up to null sets) partition $X = Y \sqcup Z$ such that $\mathcal{S} \upharpoonright Y$ is parabolic and $\mathcal{S} \upharpoonright Z$ is loxodromic. Let $\nu \in \text{Fix}(\mathcal{S} \upharpoonright Y)$ denote the unique element and let $\eta \in \text{Fix}(\mathcal{S} \upharpoonright Z)$ denote the unique element satisfying: η_z has two atoms of equal mass $1/2$ for a.e. $z \in Z$. Finally, define $\omega \in \text{Fix}(\mathcal{S})$ by $\omega_y = \nu_y$ for $y \in Y$ and $\omega_z = \eta_z$ for $z \in Z$.

Lemma 4.3. *Suppose the Main Assumption is satisfied. Let $\mathcal{S} \leq \mathcal{T} \leq \mathcal{R}$ be aperiodic hyperfinite subequivalence relations. Then the canonical \mathcal{S} -invariant field of boundary measures is the same as the canonical \mathcal{T} -invariant field of boundary measures.*

Proof. Let $X = Y \sqcup Z$ and ν, η, ω be as in Definition 10. Note that an inclusion of subequivalence relations always implies the reverse inclusion of corresponding fixed points in the space of fields of boundary measures. Therefore, $\text{Fix}(\mathcal{T} \upharpoonright Y) \subset \text{Fix}(\mathcal{S} \upharpoonright Y) = \{\nu\}$. Since

$\text{Fix}(\mathcal{T} \upharpoonright Y)$ is nonempty (Theorem 3.2), this implies $\text{Fix}(\mathcal{T} \upharpoonright Y) = \{\nu\}$. In particular, $\mathcal{T} \upharpoonright Y$ is parabolic.

Since $\mathcal{S} \upharpoonright Z$ is loxodromic and $\mathcal{S} \upharpoonright Z \leq \mathcal{T} \upharpoonright Z$, Theorem 3.5 implies $\mathcal{T} \upharpoonright Z$ is also loxodromic. To be precise, Theorem 3.5 apriori only applies to ergodic equivalence relations. However, by decomposing a nonergodic measure into its ergodic components, we see that Theorem 3.5 extends to nonergodic equivalence relations as well. So there exists a measure $\kappa \in \text{Fix}(\mathcal{T} \upharpoonright Z)$ such that for a.e. $z \in Z$, κ_z has two atoms of equal mass $1/2$. Because $\mathcal{S} \leq \mathcal{T}$, we have $\kappa \in \text{Fix}(\mathcal{S} \upharpoonright Z)$. We claim that $\kappa = \eta$. Indeed, if this is not true then $(1/2)(\kappa + \eta)_z$ has support containing more than 3 elements (for all z is a set with positive measure), contradicting Theorem 3.2. So $\kappa = \eta$ which implies the Corollary. \square

Proof of Theorem 4.2. Let \mathcal{R} be a hyperbolic equivalence relation and $\mathcal{S} \leq \mathcal{R}$ an aperiodic hyperfinite subequivalence relation. Let $\omega \in \text{Fix}(\mathcal{S})$ be the canonical \mathcal{S} -invariant field of boundary measures (Definition 10). Let \mathcal{M} be the set of all $(x, y) \in \mathcal{R}$ such that $\alpha(x, y)_* \omega_y = \omega_x$. In other words, \mathcal{M} is the stabilizer of ω in \mathcal{R} . Clearly, $\mathcal{S} \leq \mathcal{M}$. It is easily checked that \mathcal{M} is a subequivalence relation. It must be hyperfinite by Theorem 3.2.

In order to show that it is maximal, let \mathcal{K} be a measurable hyperfinite subequivalence relation with $\mathcal{M} \leq \mathcal{K} \leq \mathcal{R}$. By Lemma 4.3, $\omega \in \text{Fix}(\mathcal{K})$. The definition of \mathcal{M} now implies $\mathcal{K} = \mathcal{M}$. \square

Definition 11. Let $\mathcal{S} \leq \mathcal{R}$ be a subequivalence relation. For any $\phi \in \text{Aut}(X, \mu)$ we let

$$\phi_* \mathcal{S} = \{(\phi(x), \phi(y)) : (x, y) \in \mathcal{S}\}.$$

If $\phi \in [\mathcal{R}]$ then this is a subequivalence relation of \mathcal{R} . The **quasi-normalizer** of \mathcal{S} in \mathcal{R} is the subgroup $N_{\mathcal{R}}^q(\mathcal{S}) \leq [\mathcal{R}]$ generated by all $\phi \in [\mathcal{R}]$ such that $\phi_* \mathcal{S} \cap \mathcal{S}$ is aperiodic. The **normalizer** of \mathcal{S} in \mathcal{R} is the subgroup $N_{\mathcal{R}}(\mathcal{S}) \leq [\mathcal{R}]$ generated by all $\phi \in [\mathcal{R}]$ such that $\phi_* \mathcal{S} = \mathcal{S}$. So if \mathcal{S} is aperiodic then $N_{\mathcal{R}}(\mathcal{S}) \leq N_{\mathcal{R}}^q(\mathcal{S})$.

Definition 12. Let $\text{Aut}(X, \mu)$ denote the group of measure-preserving Borel isomorphisms of (X, μ) . Let $\mathcal{S} \leq \mathcal{R}$ be a subequivalence relation and $G \subset \text{Aut}(X, \mu)$ a subset. We let $\langle \mathcal{S}, G \rangle$ denote the smallest equivalence relation on X containing both \mathcal{S} and $\{(x, gx) : g \in G, x \in X\}$.

Theorem 4.4. *Let (X, μ) be a standard probability space and $\mathcal{R} \subset X \times X$ a discrete Borel equivalence relation such that μ is \mathcal{R} -invariant and \mathcal{R} -ergodic. If \mathcal{R} has rank 1 then for every aperiodic hyperfinite subequivalence relation $\mathcal{S} \leq \mathcal{R}$, $\langle \mathcal{S}, N_{\mathcal{R}}^q(\mathcal{S}) \rangle$ is hyperfinite.*

It is an open problem whether the converse holds. The next result is essentially the same as [JKL02, Lemmas 2.9(ix) and 2.15(ix)]. We give a proof for the reader's convenience.

Lemma 4.5. *Let (X, μ) denote a standard probability space. Let $\mathcal{R} \subset X \times X$ be a hyperfinite Borel equivalence relation on X and assume μ is \mathcal{R} -invariant. Suppose $\phi \in \text{Aut}(X, \mu)$ normalizes \mathcal{R} (this means that $x\mathcal{R}y \Rightarrow (\phi x)\mathcal{R}(\phi y)$). Then $\langle \mathcal{R}, \phi \rangle$ is hyperfinite.*

Proof. Given a discrete Borel equivalence relation \mathcal{S} on (X, μ) , we let $[\mathcal{S}]$ acts on $L^\infty(\mathcal{S})$ by

$$\theta f(x, y) := f(\theta^{-1}x, y), \quad \forall \theta \in [\mathcal{S}], f \in L^\infty(\mathcal{S}).$$

We let $L^\infty(\mathcal{S})^*$ denote the Banach dual of $L^\infty(\mathcal{S})$. We also let $[\mathcal{S}]$ act on $L^\infty(\mathcal{S})^*$ by

$$(\theta\Phi)(f) = \Phi(\theta^{-1}f), \quad \forall \theta \in [\mathcal{S}], f \in L^\infty(\mathcal{S}), \Phi \in L^\infty(\mathcal{S})^*.$$

Following [CFW81, Definition 5] a countable pmp equivalence relation \mathcal{S} on (X, μ) is called **amenable** if there exists a state $\Lambda : L^\infty(\mathcal{S}) \rightarrow \mathbb{C}$ such that Λ is $[\mathcal{S}]$ -invariant and $\Lambda(g) = \int g \, d\mu$ for all $g \in L^\infty(X)$ where we have embedded $L^\infty(X)$ into $L^\infty(\mathcal{R})$ via $g(x, y) = g(x)$ for $g \in L^\infty(X)$. By [CFW81, Theorem 10] a countable pmp equivalence relation \mathcal{S} is amenable if and only if it is hyperfinite.

Since \mathcal{R} is hyperfinite, there exists a state $\Lambda : L^\infty(\mathcal{R}) \rightarrow \mathbb{C}$ satisfying the above requirements. By viewing $\mathcal{R} \subset \langle \mathcal{R}, \phi \rangle$ we obtain the restriction map $\text{Res} : L^\infty(\langle \mathcal{R}, \phi \rangle) \rightarrow L^\infty(\mathcal{R})$. Now define $\Psi_n \in L^\infty(\langle \mathcal{R}, \phi \rangle)^*$ by

$$\Psi_n(f) = \frac{1}{n} \sum_{i=0}^n \Lambda(\text{Res}(\phi^{-i}f)).$$

Since each Ψ_n is contained in the unit ball of $L^\infty(\langle \mathcal{R}, \phi \rangle)^*$, the Banach-Alaoglu Theorem implies the existence of a weak* limit point, denoted Ψ , of $\{\Psi_n\}$. We claim that Ψ is a state verifying the properties of amenability.

First we observe that if $\theta \in [\mathcal{R}]$ then

$$\Psi_n(\theta f) = \frac{1}{n} \sum_{i=0}^n \Lambda(\text{Res}(\phi^{-i}\theta f)).$$

Since Λ is $[\mathcal{R}]$ -invariant and $\phi^{-i}\theta\phi^i \in [\mathcal{R}]$,

$$\Lambda(\text{Res } \phi^{-i}\theta\phi^i\phi^{-i}f) = \Lambda(\text{Res } \phi^{-i}f).$$

Hence $\Psi_n(\theta f) = \Psi_n(f)$ and so $\Psi(\theta f) = \Psi(f)$ for any $\theta \in [\mathcal{R}]$, $f \in L^\infty(\langle \mathcal{R}, \phi \rangle)$.

By construction, we have that $\Psi(\phi f) = \Psi(f)$ for any $f \in \langle \mathcal{R}, \phi \rangle$. So Ψ is both ϕ -invariant and $[\mathcal{R}]$ -invariant. Since ϕ and $[\mathcal{R}]$ generate $[\langle \mathcal{R}, \phi \rangle]$, this proves Ψ is $[\langle \mathcal{R}, \phi \rangle]$ -invariant.

Next, let $g \in L^\infty(X) \subset L^\infty(\langle \mathcal{R}, \phi \rangle)$. Note that $\text{Res}(\phi^{-i}g)(x, y) = g(\phi^i x) = \phi^{-i}g(x)$. So

$$\Psi_n(g) = \frac{1}{n} \sum_{i=0}^n \Lambda(\text{Res}(\phi^{-i}g)) = \frac{1}{n} \sum_{i=0}^n \Lambda(\phi^{-i}g).$$

Since $\Lambda(\phi^{-i}g) = \int \phi^{-i}g \, d\mu = \int g \, d\mu$ (since ϕ is measure-preserving), we have $\Psi_n(g) = \int g \, d\mu$ for all n . So $\Psi(g) = \int g \, d\mu$. Since g is arbitrary, this proves that Ψ is an invariant mean. Thus $\langle \mathcal{R}, \phi \rangle$ is amenable. \square

Proof of Theorem 4.4. Suppose \mathcal{R} has rank 1 and $\mathcal{S} \leq \mathcal{R}$ is aperiodic and hyperfinite. Then \mathcal{S} is contained in a unique maximal hyperfinite subequivalence relation $\mathcal{M} \leq \mathcal{R}$. Let $\phi \in [\mathcal{R}]$ quasi-normalize \mathcal{S} . Since $\mathcal{S} \cap \phi_*\mathcal{S}$ is aperiodic it must be that $\mathcal{M} \cap \phi_*\mathcal{M}$ is aperiodic too. Because \mathcal{M} is the unique maximal hyperfinite subequivalence relation containing $\mathcal{M} \cap \phi_*\mathcal{M}$ and $\phi_*\mathcal{M}$ is also a maximal hyperfinite subequivalence relation, it follows that $\mathcal{M} = \phi_*\mathcal{M}$. Thus $N_{\mathcal{R}}^q(\mathcal{S})$ normalizes \mathcal{M} . The previous lemma now implies the theorem. \square

We now show that various kinds of equivalence relations are higher rank and in particular, cannot satisfy the Main Assumption (which implies that they do not admit hyperbolic graphings as in Example 1 and cannot be treeable).

Corollary 4.6. *Suppose \mathcal{R} is the orbit-equivalence relation for an ergodic pmp essentially free action of $SL(n, \mathbb{Z})$ for some $n \geq 3$. Then \mathcal{R} has higher rank.*

Proof. For $i, j \in \{1, \dots, n\}$ with $i \neq j$ let $E_{ij} \in SL(n, \mathbb{Z})$ denote the matrix with 1's on the diagonal, a 1 in the (i, j) -th entry and 0's everywhere else. Let $\mathcal{R}_{ij} \leq \mathcal{R}$ denote the orbit subequivalence relation generated by E_{ij} : $\mathcal{R}_{ij} = \{(x, E_{ij}^m x) : x \in X, m \in \mathbb{Z}\}$. Each \mathcal{R}_{ij} is aperiodic and hyperfinite.

To obtain a contradiction, suppose \mathcal{R} has rank 1 and let $\mathcal{M} \leq \mathcal{R}$ denote the unique maximal hyperfinite subequivalence relation containing \mathcal{R}_{12} . Because E_{12} commutes with E_{1k} ($2 \leq k \leq n$), it follows the subequivalence relation $\mathcal{R}_{12} \vee \mathcal{R}_{1k}$ is also hyperfinite and therefore $\mathcal{R}_{12} \vee \mathcal{R}_{1k} \leq \mathcal{M}$. Since E_{1k} commutes with E_{jk} (for $j \neq k$) it follows that $\mathcal{R}_{1k} \vee \mathcal{R}_{jk}$ is hyperfinite and therefore $\mathcal{R}_{jk} \leq \mathcal{M}$ for $j \neq k$. Since $\{E_{jk}\}_{j \neq k}$ generates $SL(n, \mathbb{Z})$, it follows that $\mathcal{M} = \mathcal{R}$. This contradicts the fact that, since $SL(n, \mathbb{Z})$ is non-amenable, \mathcal{R} is non-hyperfinite. \square

Corollary 4.7. *Let A, B be countably infinite groups, $G = A \times B$, $G \curvearrowright (X, \mu)$ an essentially free ergodic pmp action and $\mathcal{R} \subset X \times X$ the orbit-equivalence relation. If B is non-amenable and A contains an infinite amenable subgroup $A' \leq A$ then \mathcal{R} has higher rank.*

Proof. Let $\mathcal{S}, \mathcal{R}_B$ denote the orbit subequivalence relations generated by A', B respectively. If we view B as a subgroup of $[\mathcal{R}]$ then B is contained in the normalizer of \mathcal{S} . Thus the subequivalence relation generated by B and \mathcal{S} contains \mathcal{R}_B and is therefore non-hyperfinite (since B is non-amenable). By Theorem 4.4, this implies \mathcal{R} has higher rank. \square

It is an open problem whether the hypothesis that A contain an infinite amenable subgroup can be removed.

5 Loxodromic elements

The purpose of this section is to prove:

Theorem 5.1. *Suppose the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite. Then there exists an ergodic loxodromic element $\phi \in [\mathcal{R}]$. Moreover, we can choose ϕ such that for a.e. $x \in X$ the path in \mathcal{H}_x obtained by concatenating geodesic segments $[\sigma_x(\phi^n(x)), \sigma_x(\phi^{n+1}(x))]$ is a quasi-geodesic.*

For the rest of this section, we assume the hypotheses of Theorem 5.1 above.

5.1 Proof sketch

To prove Theorem 5.1 we will build the loxodromic element $\phi \in [\mathcal{R}]$ as an increasing limit of partial transformations. To be precise, a **partial transformation** of \mathcal{R} is a measure-space isomorphism

$$\phi : \text{dom}(\phi) \rightarrow \text{rng}(\phi)$$

where $\text{dom}(\phi), \text{rng}(\phi) \subset X$ and the graph of ϕ is contained in \mathcal{R} . We let $[\![\mathcal{R}]\!]$ denote the set of all such partial transformations. As per our usual convention, we identify two partial transformations that agree almost everywhere.

Definition 13. Let $F \subset \mathcal{R}$ be measurable. A **directed matching** in F is a partial transformation $\phi \in [\![\mathcal{R}]\!]$ such that the graph of ϕ is contained in F and $\text{dom}(\phi) \cap \text{rng}(\phi)$ has measure zero. It is **perfect** if $X = \text{dom}(\phi) \cup \text{rng}(\phi)$ (modulo measure zero sets).

For example, if ϕ_1, ϕ_2 are two perfect directed matchings such that $\text{rng}(\phi_1) = \text{dom}(\phi_2)$ and $\text{rng}(\phi_2) = \text{dom}(\phi_1)$ then the composition $\phi_2 \circ \phi_1 \in [\mathcal{R}]$.

By Lemma 3.7 there are constants $r, s > 0$ such that to prove the existence of a not-necessarily-ergodic loxodromic element $\phi \in [\mathcal{R}]$ it suffices to construct two directed perfect matchings ϕ_1, ϕ_2 such that

- $\text{rng}(\phi_1) = \text{dom}(\phi_2), \text{rng}(\phi_2) = \text{dom}(\phi_1),$
- $d_\sigma(x, \phi_i(x)) \geq r$ for a.e. x and $i = 1, 2,$
- $(x|\phi_2 \circ \phi_1 x)_{\phi_1 x}^\sigma \leq s$ for a.e. x

where

$$(x|z)_y^\sigma = \frac{1}{2} \left(d_\sigma(x, y) + d_\sigma(y, z) - d_\sigma(x, z) \right).$$

(Recall that $d_\sigma(x, y) = d(\sigma(x), \alpha(x, y)\sigma(y))$.) Then $\phi := \phi_2 \circ \phi_1$ is loxodromic because the path obtained by concatenating geodesic segments $[\sigma_x(\phi^i(x)), \sigma_x(\phi^{i+1}x)]$ is a quasi-geodesic.

To obtain such matchings, we will use the hypotheses of “extreme expansivity” (explained next). This hypothesis is stronger than the ones studied in [LN11, CL12] and the proofs are simpler although less constructive. The construction of an *ergodic* loxodromic element is a bit more involved since it requires that we control the averages $\frac{1}{n} \sum_{i=1}^n f(\phi^i(x))$ for test functions $f \in L^2(X, \mu)$. However, the general principle is the same.

5.2 Matchings

Definition 14. A measurable set $F \subset \mathcal{R}$ is **extremely expansive** for every pair of non-null sets $A, B \subset X$, $F \cap (A \times B)$ is non-null (with respect to the usual measure $\hat{\mu}$ on \mathcal{R} , see §B).

The usefulness of this condition is that it implies the existence of perfect matchings:

Proposition 5.2. *If $F \subset \mathcal{R}$ is extremely expansive then for every pair of non-null disjoint sets $A, B \subset X$ with $\mu(A) = \mu(B)$ there exist a directed matching $\phi \in \llbracket \mathcal{R} \rrbracket$ with $\text{graph}(\phi) \subset F \cap (A \times B)$ and $\text{dom}(\phi) = A$, $\text{rng}(\phi) = B$. In particular, there exists a perfect directed matching.*

To prove this, we need a short lemma first.

Lemma 5.3. *If $F \subset \mathcal{R}$ is extremely expansive and $A, B \subset X$ are disjoint sets each with positive measure then there exists a directed matching $\phi \in \llbracket \mathcal{R} \rrbracket$ with $\mu(\text{dom}(\phi)) > 0$, $\text{graph}(\phi) \subset F \cap (A \times B)$.*

Proof. By [FM77, Theorem 1], there exists a countable subset $\mathcal{F} \subset [\mathcal{R}]$ such that for a.e. $(x, y) \in \mathcal{R}$ there exists $f \in \mathcal{F}$ with $f(x) = y$. Because F is extremely expansive, there exists $f \in \mathcal{F}$ such that

$$\text{graph}(f) \cap F \cap (A \times B)$$

is non-null. So we define ϕ so that its graph equals $\text{graph}(f) \cap F \cap (A \times B)$. □

Proof of Proposition 5.2. Let \mathcal{C} be the set of all $\phi \in \llbracket \mathcal{R} \rrbracket$ such that $\text{graph}(\phi) \subset F \cap (A \times B)$. There is a natural partial order on \mathcal{C} given by $\phi \leq \psi$ if $\text{dom}(\phi) \subset \text{dom}(\psi) \pmod{\mu}$ and $\psi(x) = \phi(x)$ for a.e. $x \in \text{dom}(\phi)$.

By a standard measure exhaustion argument (as in Lemma 4.1), there exists a maximal element $\phi \in \mathcal{C}$. If $\text{dom}(\phi)$ is not co-null in A then the previous lemma implies the existence of a measure-space isomorphism

$$\psi : (A \setminus \text{dom}(\phi)) \rightarrow (B \setminus \text{rng}(\phi))$$

with graph contained in F . Setting $\Phi = \phi \sqcup \psi$ yields an element of \mathcal{C} that is greater than ϕ in the partial ordering, contradicting the maximality of ϕ . So $\text{dom}(\phi) = A$ up to measure-zero. Since ϕ is measure-preserving, $\text{rng}(\phi) = B$ up to measure zero. In particular, if $A \cup B = X$ then ϕ is perfect. □

5.3 Matchings in \mathcal{R} via hyperbolic geometry

For any $t > 0$ let X_t be the set of all $x \in X$ such that there exist distinct elements $\xi, \eta \in \partial\mathcal{H}_x$ with

$$(\xi|\eta)_{\sigma(x)} < t.$$

Because $X = \cup_{t>0} X_t$, there exists $s > \delta$ such that $\mu(X_{s-\delta}) > 0$. Later, we will use this value of s in Lemma 3.7. The next proposition is the key tool for proving the existence of directed matchings satisfying geometric constraints.

Proposition 5.4. *Let $\psi : X_{s-\delta} \rightarrow X_{s-\delta}$ be a Borel map with graph in \mathcal{R} (we do not require that ψ is invertible). Also let $r > 0$. Let F be the set of all $(x, y) \in \mathcal{R} \upharpoonright X_{s-\delta}$ such that*

$$d_\sigma(x, y) > r, \quad (y|\psi x)_x^\sigma < s, \text{ and } (x|\psi y)_y^\sigma < s.$$

Then F is extremely expansive in $Y \times Y$ where $Y = X_{s-\delta}$.

Proof. Let $A, B \subset Y$ be non-null sets. It suffices to show that $F \cap (A \times B)$ is non-null.

Given $(x, y) \in \mathcal{R} \upharpoonright Y$ and $s > 0$, let $\text{Shadow}_s(x, y) \subset \overline{\mathcal{H}}_x$ be the set of all $p \in \overline{\mathcal{H}}_x$ such that

$$(\sigma_x(y)|p)_{\sigma(x)} \geq s.$$

This is a closed subset.

Claim 1. For a.e. $x \in Y$ and $y \in A$, the limit set \mathcal{L}_x is not contained in $\alpha(x, y)\text{Shadow}_s(y, \psi y)$.

Proof of Claim 1. Let $\xi, \eta \in \text{Shadow}_s(y, \psi y)$ be distinct elements. By Gromov's inequality (see §A),

$$(\xi|\eta)_{\sigma(y)} \geq \min\{(\xi|\psi y)_{\sigma(y)}, (\psi y|\eta)_{\sigma(y)}\} - \delta \geq s - \delta.$$

Because $y \in Y$ there exist distinct elements $\xi', \eta' \in \mathcal{L}_y$ such that

$$s - \delta > (\xi'|\eta')_{\sigma(y)}.$$

Therefore, ξ', η' cannot both be in $\text{Shadow}_s(y, \psi y)$. Thus \mathcal{L}_x is not contained in $\alpha(x, y)\text{Shadow}_s(y, \psi y)$. □

Claim 2. For $x \in Y$, let

$$K_x := \bigcap_{y \in A \cap [x]_{\mathcal{R}}} \alpha(x, y) \text{Shadow}_s(y, \psi y).$$

Then $K_x = \emptyset$ for a.e. x .

Proof of Claim 2. Note K_x is closed and invariant (in the sense that $\alpha(x, y)K_y = K_x$). By Claim 1 and Lemmas 2.6 and 2.8, $\text{Hull}(K_x) \cap \mathcal{H}_x = \emptyset$ for a.e. x . So either $K_x = \emptyset$ for a.e. x or $\text{Hull}(K_x)$ consists of a single point in $\partial \mathcal{H}_x$ for a.e. x . However the latter possibility implies that \mathcal{R} is hyperfinite (Theorem 3.2) contradicting our hypotheses. \square

By symmetry, the claims above also hold with B in place of A . Because $\overline{\mathcal{O}_x^\sigma} \subset \overline{\mathcal{H}_x}$ is compact, Claim 2 implies that for a.e. x there exist finite sets $A_x \subset A \cap [x]$ and $B_x \subset B \cap [x]$ such that

$$\overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in A_x} \alpha(x, y) \text{Shadow}_s(y, \psi y) = \overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in B_x} \alpha(x, y) \text{Shadow}_s(y, \psi y) = \emptyset.$$

For $D > 0$, let X_D be the set of all $x \in Y$ such that there exist finite sets $S \subset A \cap [x], T \subset B \cap [x]$ satisfying

- $\text{diam}(S) \leq D$ and $\text{diam}(T) \leq D$ (where diameter is computed with respect to d_σ),

•

$$\overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in T} \alpha(x, y) \text{Shadow}_s(y, \psi y) = \overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in S} \alpha(x, y) \text{Shadow}_s(y, \psi y) = \emptyset.$$

Because $Y = \cup_{D>0} X_D$, there exists some $D > 0$ such that X_D has positive measure. Since X_D is \mathcal{R} -invariant, this implies $Y = X_D$ (up to measure zero).

Let I_x be the closure in $\overline{\mathcal{H}_x}$ of the set of all $\sigma_x(y)$ such that y is in a finite subset $S \subset B \cap [x]$ of diameter $\leq D$ satisfying

$$\overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in S} \alpha(x, y) \text{Shadow}_s(y, \psi y) = \emptyset.$$

Because $x \mapsto \partial I_x$ is \mathcal{R} -invariant (where $\partial I_x = I_x \cap \mathcal{L}_x$) Theorem 2.9 implies $\partial I_x = \mathcal{L}_x$ for a.e. x .

So for a.e. $a \in A$ and every $\xi \in \mathcal{L}_a$ there exists a sequence $\{S_i\}$ of finite sets $S_i \subset B \cap [a]$ satisfying $\text{diam}(S_i) \leq D$ and

$$\overline{\mathcal{O}_x^\sigma} \cap \bigcap_{y \in S_i} \alpha(a, y) \text{Shadow}_s(y, \psi y) = \emptyset$$

while $\lim_i \sigma_a(S_i) = \xi$. By Claim 1 we may choose $\xi \in \mathcal{L}_a \setminus \text{Shadow}_s(a, \psi a)$. So there must exist $b \in B$ such that

- $\sigma(b) \notin \alpha(b, a) \text{Shadow}_s(a, \psi a)$ and $\sigma(a) \notin \alpha(a, b) \text{Shadow}_s(b, \psi b)$
- $d_\sigma(a, b) > r$.

Equivalently $(a, b) \in F$. Note that F is a Borel set and we just proved that almost every vertical section of $F \cap (A \times B)$ is nonempty. So $\widehat{\mu}(F \cap A \times B) \geq \mu(A) > 0$. This proves F is extremely expansive. \square

Proof of Theorem 5.1. As above, let $s > 0$ be such that $\mu(X_{s-\delta}) > 0$. Let $r > 0$ be as in Lemma 3.7. To simplify notation, let $Y = X_{s-\delta}$ and μ_Y be the probability measure on Y obtained by restricting μ and normalizing. We will first show that there exists an ergodic loxodromic element $\Phi \in [\mathcal{R} \upharpoonright Y]$.

Recall the definition of partial transformation from §5.1. There is a natural partial order on $[\mathcal{R} \upharpoonright Y]$ given by $\phi \leq \psi$ if $\text{dom}(\phi) \subset \text{dom}(\psi) \pmod{\mu}$ and $\psi(x) = \phi(x)$ for a.e. $x \in \text{dom}(\phi)$ (this is just containment of graphs mod $\widehat{\mu}$).

Let $\mathcal{F}_\infty \subset L^2(Y)$ be the set of all functions f such that

- $0 \leq f \leq 1$,
- the map $t \mapsto \mu(\{x \in Y : t > f(x)\})$ is strictly increasing for $0 < t < 1$.

Then \mathcal{F}_∞ has dense linear span in $L^2(Y)$. So there exists an increasing sequence $\{\mathcal{F}_i\}_{i=1}^\infty$ of finite subsets $\mathcal{F}_i \subset \mathcal{F}_\infty$ whose union has dense linear span in $L^2(Y)$. We let $\|\cdot\|$ denote the L^2 -norm on $L^2(Y)$.

By induction we will construct an increasing sequence $\{\phi_i\}_{i \in \mathbb{N}} \subset [\mathcal{R} \upharpoonright Y]$ satisfying the following.

0. $Y = \text{dom}(\phi_1) \cup \text{rng}(\phi_1)$ (modulo null sets).

1. If $\mathcal{S}_i \leq (\mathcal{R} \upharpoonright Y)$ is the subequivalence relation generated by ϕ_i (so $x\mathcal{S}_i y \Leftrightarrow \exists n \in \mathbb{Z}$ such that $\phi_i^n(x) = y$) then almost every \mathcal{S}_i -equivalence class is finite.
2. For a.e. x , $[x]_{\mathcal{S}_{i-1}} \subsetneq [x]_{\mathcal{S}_i}$ (if $i > 1$).
3. $d_\sigma(x, \phi_i(x)) \geq r$ for a.e. $x \in \text{dom}(\phi_i)$.
4. $(x|\phi_i^2(x))_{\phi(x)}^\sigma \leq s$ for a.e. $x \in \text{dom}(\phi_i^2)$.
5. If $A_i : L^2(Y) \rightarrow L^2(Y)$ is the averaging operator defined by

$$A_i(f)(x) = \frac{1}{|[x]_{\mathcal{S}_i}|} \sum_{y \in [x]_{\mathcal{S}_i}} f(y)$$

then $\|A_i(f) - \int_Y f d\mu_Y\| \leq 2\|f\|/i$ for all $f \in \mathcal{F}_i$.

Suppose for the moment that we have constructed such a sequence. Define $\phi \in \llbracket \mathcal{R} \upharpoonright Y \rrbracket$ by $\phi(x) = \phi_i(x)$ for $x \in \text{dom}(\phi_i)$. Because the ϕ_i 's are increasing this is well-defined. By item (2), almost every orbit of ϕ is infinite. Therefore $\text{dom}(\phi) = \text{rng}(\phi)$ modulo null sets. By item (0), $\text{dom}(\phi) = Y \pmod{\mu}$. So $\phi \in [\mathcal{R} \upharpoonright Y]$.

Let $\mathcal{S}_\infty = \cup_i \mathcal{S}_i$. Observe that \mathcal{S}_∞ is the sub-equivalence relation generated by ϕ . Recall that a subset $Z \subset Y$ is \mathcal{S}_∞ -saturated if it is a union of \mathcal{S}_∞ -classes. By the Martingale Convergence Theorem, for every $f \in \cup_i \mathcal{F}_i$, $A_i(f)$ converges to $\mathbb{E}[f|\mathcal{S}_\infty]$ which denotes the conditional expectation of f on the sigma-algebra of \mathcal{S}_∞ -saturated measurable sets. Item (5) now implies that for any $f \in \cup_i \mathcal{F}_i$, $\mathbb{E}[f|\mathcal{S}_\infty] = \int f d\mu_Y$ is constant. Since $\cup_i \mathcal{F}_i$ is dense in $L^2(Y)$, it follows by continuity that $f \mapsto \mathbb{E}[f|\mathcal{S}_\infty]$ is projection onto the constants for all $f \in L^2(Y)$. Therefore \mathcal{S}_∞ is ergodic. Because ϕ generates \mathcal{S} , ϕ is ergodic.

Conditions (3,4) and Lemma 3.7 imply that for a.e. x the orbit $\{\phi^n(x)\}_{n \in \mathbb{Z}}$ is a quasi-geodesic with respect to d_σ . Thus ϕ is loxodromic as required.

We will construct the ϕ_i 's by induction. Here is the base case: let F_1 be the set of all $(x, y) \in \mathcal{R} \upharpoonright Y$ such that $d_\sigma(x, y) \geq r$. Clearly, F_1 is extremely expansive. So Proposition 5.2 implies the existence of a perfect directed matching $\phi_1 \in \llbracket \mathcal{R} \upharpoonright Y \rrbracket$ whose graph is contained in F_1 ; that is $d_\sigma(x, \phi_1(x)) \geq r$ for a.e. x .

Observe that ϕ_1 satisfies (0-5). It satisfies item (0) since ϕ_1 is perfect. The equivalence class of x generated by ϕ_1 is just $[x]_{\mathcal{S}_1} = \{x, \phi_1(x)\}$. This proves (1). Item (3) holds by design.

Items (2,4) hold vacuously since $\text{dom}(\phi_1^2) = \emptyset$. Item (5) holds because $\|A_i(f)\| \leq \|f\|$ for any $f \in L^2(Y)$ since ϕ_1 is measure-preserving.

Now suppose that ϕ_1, \dots, ϕ_i have been constructed so that items (1-5) above hold and ϕ_1 is a perfect matching. We now turn to constructing ϕ_{i+1} . Because ϕ_1 is a perfect matching and $\phi_1 \leq \phi_i$ it follows that $\text{dom}(\phi_i) \cup \text{rng}(\phi_i) = Y$ (up to null sets). So we have a natural partition

$$Y = (Y \setminus \text{rng}(\phi_i)) \sqcup (\text{dom}(\phi_i) \cap \text{rng}(\phi_i)) \sqcup (Y \setminus \text{dom}(\phi_i)).$$

Because every $f \in \mathcal{F}_{i+1}$ is bounded there exists a partition \mathcal{P} of $Y \setminus \text{rng}(\phi_i)$ such that for each $f \in \mathcal{F}_{i+1}$, $P \in \mathcal{P}$ and $x, y \in P$

$$|A_i f(x) - A_i f(y)| \leq \|f\|/(i+1).$$

By further refining \mathcal{P} and perturbing it slightly (using the fact that the map $t \mapsto \mu(\{x \in Y : t > f(x)\})$ is strictly increasing for $0 < t < 1$) we may assume that $\mathcal{P} = \{P_1, \dots, P_n\}$ for some $n \geq 2$ and $\mu_Y(P_i) = \mu_Y(P_j)$ for all i, j .

Let P'_j be the set of all $x \in Y \setminus \text{dom}(\phi_i)$ such that there exists $y \in P_j$ with $\phi_i^m(y) = x$ for some $m \geq 1$. So $\mathcal{P}' = \{P'_1, \dots, P'_n\}$ is a partition of $Y \setminus \text{dom}(\phi_i)$ and $\mu_Y(P_j) = \mu_Y(P'_j)$ for all j .

Let $\psi : X_{s-\delta} \rightarrow X_{s-\delta}$ be the map

$$\psi(x) = \begin{cases} \phi_i(x) & x \in \text{dom}(\phi_i) \\ \phi_i^{-1}(x) & x \notin \text{dom}(\phi_i) \end{cases}$$

Let F be the set of all pairs $(x, y) \in \mathcal{R} \upharpoonright Y$ such that

$$d_\sigma(x, y) \geq r, \quad (y|\psi x)_x^\sigma \leq s, \quad (x|\psi y)_y^\sigma \leq s.$$

By Proposition 5.4, F is extremely expansive. By Proposition 5.2, there exists a partial transformation $\psi_j \in \llbracket \mathcal{R} \rrbracket$ such that $\text{dom}(\psi_j) = P'_j$, $\text{rng}(\psi_j) = P_{j+1}$ and $\text{graph}(\psi_j) \subset F \cap (P'_j \times P_{j+1})$ ($1 \leq j < n$).

We define ϕ_{i+1} by: $\phi_{i+1}(x) = \phi_i(x)$ if $x \in \text{dom}(\phi_i)$ and $\phi_{i+1}(x) = \psi_j(x)$ if $x \in P'_j$ for some $1 \leq j < n$. Because ϕ_{i+1} is not defined if $x \in P'_n$ each \mathcal{S}_{i+1} -class is finite. This proves (1). Items (2) and (3) are immediate.

To check item (4) let $x \in \text{dom}(\phi_{i+1}^2)$. If $x \in \text{dom}(\phi_i^2)$ then (4) holds by the induction hypothesis. If $x \in \text{dom}(\phi_i)$ but not in $\text{dom}(\phi_i^2)$ then $\phi_{i+1}(x) = \phi_i(x)$ and $\phi_{i+1}^2(x) = \psi_j \phi_i(x)$ for some j . Since $\text{graph}(\psi_j) \subset F$, $(\phi_{i+1}x, \phi_{i+1}^2x) \in F$ which implies

$$(\phi_{i+1}^2x | \psi \phi_{i+1}x)_{\phi_{i+1}x}^\sigma \leq s.$$

Since $\psi \phi_{i+1}x = \psi \phi_i x = x$ this shows

$$(\phi_{i+1}^2x | x)_{\phi_{i+1}x}^\sigma \leq s.$$

The last case, when $x \notin \text{dom}(\phi_i)$, is similar.

Next we check item (5). Note that for each $1 \leq j \leq n$, every \mathcal{S}_{i+1} -class contains exactly one \mathcal{S}_i -class that nontrivially intersects P_j . Also $A_{i+1}(f)(x)$ is a convex sum of $A_i(f)(y)$ where y varies over any set of representatives of the \mathcal{S}_i -classes in $[x]_{\mathcal{S}_{i+1}}$. Because $|A_i f(x) - A_i f(y)| \leq \|f\|/(i+1)$ for each $x, y \in P_j$ and $f \in \mathcal{F}_{i+1}$, it follows that $|A_{i+1}(f)(x) - A_{i+1}(f)(y)| \leq \|f\|/(i+1)$ for each $x, y \in Y$. This implies

$$\left| A_{i+1}(f)(x) - \int_Y f \, d\mu_Y \right| = \left| A_{i+1}(f)(x) - \int A_{i+1}(f) \, d\mu_Y \right| \leq \|f\|/(i+1).$$

This proves the induction step. As explained above, this shows the existence of an ergodic loxodromic element $\phi \in [\mathcal{R} \upharpoonright Y]$.

It is left to construct an ergodic loxodromic subequivalence relation of \mathcal{R} (as opposed to $\mathcal{R} \upharpoonright Y$). For this purpose, let $\Psi : X \rightarrow Y$ be any measurable map with graph contained in \mathcal{R} satisfying $\Psi(x) = x$ for all $x \in Y$. Define $\mathcal{S} \leq \mathcal{R}$ by: $x \mathcal{S} y \Leftrightarrow \Psi(x)$ and $\Psi(y)$ are in the same ϕ orbit. Note that $\mathcal{S} \upharpoonright Y$ is the subequivalence relation generated by ϕ . It follows that \mathcal{S} is ergodic. Let $\eta \in \text{Fix}(\phi)$ and define $\nu \in \text{Fix}(\mathcal{S})$ by $\nu_x = \alpha(x, y)_* \eta_y$ for any $y \in Y$ with $(x, y) \in \mathcal{S}$ (for example, $y = \Psi(x)$). Because η is fixed by ϕ , ν is fixed by \mathcal{S} . It follows that, for a.e. $x \in X$, the support of ν_x contains 2 elements. So \mathcal{S} is loxodromic.

It is a well-known fact that any ergodic equivalence relation contains an ergodic element in its full group [Kec10, Theorem 3.5]. So there exists an ergodic $\psi \in [\mathcal{S}]$. Because $\psi \in [\mathcal{S}]$, $\text{Fix}(\mathcal{S}) \subset \text{Fix}(\psi)$. Therefore ψ is also loxodromic.

□

6 Tits' alternative

The goal of this section is to prove Theorem 1.2. We do this by first constructing an \mathbb{F}_2 -action on a positive measure subset $Y \subset X$ with orbits contained in $\mathcal{R} \upharpoonright Y$. In order to prove that the constructed action is essentially free we will use the following criterion:

Lemma 6.1. *Let (\mathcal{H}, d) denote a complete δ -hyperbolic geodesic metric space. Let T_4 denote the 4-regular tree. Then for every $s > 0$ there exists an $r > 0$ (depending only on s and δ) such that if $\phi : T_4 \rightarrow \mathcal{H}$ is any map satisfying*

- *for every edge $\{v, w\}$ of T_4 , $d(\phi(v), \phi(w)) \geq r$,*
- *for any three distinct vertices $u, v, w \in V(T_4)$ such that $\{u, v\}, \{v, w\}$ are edges of T_4 ,*

$$(\phi(u)|\phi(w))_{\phi(v)} \leq s$$

then ϕ is injective. Moreover, if we consider T_4 to be a metric tree in which each edge $\{v, w\}$ has length equal to $d(\phi(v), \phi(w))$ and ϕ maps the edge from v to w isometrically onto a geodesic segment $[\phi(v), \phi(w)]$, then ϕ is a quasi-isometric embedding.

Proof. This follows from Lemma 3.7. □

The next lemma proves Theorem 1.2 up to a compression of \mathcal{R} .

Lemma 6.2. *Suppose the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite. Then there exists a subset $Y \subset X$ with $\mu(Y) > 0$ and an essentially free ergodic action $\mathbb{F}_2 \curvearrowright Y$ of the free group of rank 2 whose orbits are contained in $\mathcal{R} \upharpoonright Y$. In fact, we obtain a subequivalence relation $\mathcal{S} \leq \mathcal{R} \upharpoonright Y$ and a treeing \mathcal{T} of \mathcal{S} such that*

- *for a.e. $y \in Y$, \mathcal{T}_y is a 4-regular tree (so \mathcal{S} is treeable)*
- *if we consider \mathcal{T}_y to be a metric tree such that each edge $\{w, z\}$ has length $d(\sigma_y(w), \sigma_y(z))$ then σ_y gives a quasi-isometric embedding of \mathcal{T}_y into the fiber \mathcal{H}_y .*

Proof. By Theorem 5.1 there exists an ergodic loxodromic element $f \in [\mathcal{R}]$ such that for a.e. $x \in X$, the path γ_x obtained by concatenating consecutive geodesic segments

$[\sigma_x(f^n x), \sigma_x(f^{n+1} x)]$ is a quasi-geodesic. Let $\mathcal{L}_x(f) \subset \partial \mathcal{H}_x$ denote the limit set of the orbit $\{\sigma_x(f^n x)\}_{n \in \mathbb{Z}}$. Because γ_x is a quasi-geodesic, the limit points

$$f^{-\infty}(x) := \lim_{n \rightarrow -\infty} \sigma_x(f^n(x)), \quad f^{+\infty}(x) := \lim_{n \rightarrow \infty} \sigma_x(f^n(x))$$

exist and $\mathcal{L}_x(f) = \{f^{-\infty}(x), f^{+\infty}(x)\}$. Moreover, if ξ_x^-, ξ_x^+ are the Dirac measures concentrated on $f^{-\infty}(x), f^{+\infty}(x)$ respectively then both ξ^- and ξ^+ are invariant fields of boundary measures for the subequivalence relation $\mathcal{R}_f \leq \mathcal{R}$ generated by f .

Claim. There exists a loxodromic element $g \in [\mathcal{R}]$ such that $\mathcal{L}_x(f) \cap \mathcal{L}_x(g) = \emptyset$ for a.e. x . In fact, we can choose g to be a conjugate of f in $[\mathcal{R}]$.

Proof. Let $\nu \in \text{Fix}(f)$ be the canonical f -invariant field of boundary measures (so $\nu_x = \xi_x^-/2 + \xi_x^+/2$).

To obtain a contradiction, suppose the claim is false. Let $h \in [\mathcal{R}]$ be arbitrary. Since $\text{Fix}(hfh^{-1}) = h\text{Fix}(f)$ and $h\nu \in \text{Fix}(hfh^{-1})$ is such that $(h\nu)_x$ is supported on two elements, it must be that hfh^{-1} is loxodromic. Moreover, $h\nu$ is the canonical hfh^{-1} -invariant field of boundary measures.

Let \mathcal{S} denote the subequivalence relation generated by f and hfh^{-1} . Then \mathcal{S} fixes $\mathcal{L}(f) \cap \mathcal{L}(hfh^{-1})$. Since we are assuming $\mathcal{L}_x(f) \cap \mathcal{L}_x(hfh^{-1}) \neq \emptyset$, Theorem 3.2 implies \mathcal{S} is hyperfinite. So Lemma 4.3 implies ν and $h\nu$ are both the canonical \mathcal{S} -invariant field of boundary measures. In particular, $h\nu = \nu$. Since $h \in [\mathcal{R}]$ is arbitrary, $\nu \in \text{Fix}(\mathcal{R})$. By Theorem 3.2 this contradicts the assumption that \mathcal{R} is not hyperfinite. □

Let $g = hfh^{-1}$ be a loxodromic element with $\mathcal{L}_x(f) \cap \mathcal{L}_x(g) = \emptyset$. Because $f^{-\infty} : X \rightarrow \mathcal{L}$ is an f -invariant section, $hf^{-\infty}$ defined by $(hf^{-\infty})_x := \alpha(x, h^{-1}x)f_{h^{-1}x}^{-\infty}$ is an hfh^{-1} -section. Similarly, $hf^{+\infty}$ is an hfh^{-1} section.

For $t > 0$, let X_t be the set of all $x \in X$ such that

1. $(f^\epsilon(x)|hf^\eta(x))_x^\sigma \leq t \quad \forall \epsilon, \eta \in \{-\infty, +\infty\}$;
2. if γ is any geodesic with endpoints in $\mathcal{L}_x(f)$ then $d(\sigma(x), \gamma) \leq t$;
3. if γ is any geodesic with endpoints in $\mathcal{L}_x(g)$ then $d(\sigma(x), \gamma) \leq t$.

Then $X = \cup_{s>0} X_s$. So there exists $t > 0$ with $\mu(X_t) > 0$.

Given $u, v, w \in \overline{\mathcal{H}}_x$ and a geodesic γ we say that v lies **between** u and w with respect to γ if: for every triple $u', v', w' \in \gamma$ of points such that u' is a closest point to u on γ (and similarly with v', w') we have that u' and w' are in different components of $\gamma - \{v'\}$.

By continuity and the choice of f there exists a number $N > 0$ satisfying the following.

1. If $u, v, w \in \mathcal{H}_x$ are t -close to a geodesic $\gamma \subset \mathcal{H}_x$, $d(u, v) \geq N, d(v, w) \geq N$ and v lies between u and w with respect to γ then $(u|w)_v \leq t + 1$.
2. If $u, v, w \in \mathcal{H}_x$ and $\gamma_1, \gamma_2 \subset \mathcal{H}_x$ are geodesics such that
 - $(\gamma_1^\pm | \gamma_2^\pm)_v \leq t$ (where $\gamma_i^\pm \subset \partial \mathcal{H}_x$ are the endpoints of γ_i),
 - $d(u, \gamma_1) \leq t, d(v, \gamma_1) \leq t, d(v, \gamma_2) \leq t, d(w, \gamma_2) \leq t,$
 - $d(u, v) \geq N, d(v, w) \geq N$

then $(u|w)_v \leq t + 1$.

3. If $n > 0$ and $d(\sigma(x), \sigma_x(f^n x)) \geq N$ then $\sigma_x(f^n x)$ lies between $\sigma(x)$ and $\sigma^+(x)$ with respect to any geodesic with endpoints in $\{f^{-\infty}(x), f^{+\infty}(x)\}$.

Let $s = t + 1$. Let r be as in Lemma 6.1. We assume without loss of generality that $r > N$.

Let G be the set of all pairs $(x, y) \in \mathcal{R} \upharpoonright X_t$ such that $d_\sigma(x, y) \leq r$. Since σ is proper, G is a locally finite in the sense that for a.e. $x \in X$ there are only finitely many y with $(x, y) \in G$ (equivalently $(y, x) \in G$). By [KST99, Proposition 4.5] there exists a proper Borel vertex coloring $K : X_t \rightarrow \mathbb{N}$ of G . In other words, if $(x, y) \in G$ then $K(x) \neq K(y)$. It follows that, for some $n \in \mathbb{N}$, $K^{-1}(n)$ has positive measure. Let $X'_t = K^{-1}(n)$. The relevant properties of X'_t are: $X'_t \subset X_t$ and if $x \neq y \in X'_t$ and $x \mathcal{R} y$ then $d_\sigma(x, y) > r$.

Let $f_0 \in [\mathcal{R} \upharpoonright X'_t]$ be the first-return time map of f to X'_t . Precisely, $f_0(x) = f^n(x)$ where $n > 0$ is the smallest positive integer such that $f^n(x) \in X'_t$. Because f is ergodic, f_0 is also ergodic.

Define $g_0 \in [\mathcal{R} \upharpoonright X'_t]$ by: $g_0(x) = g^n(x)$ where $n \geq 1$ is the smallest positive integer such that $g^n(x) \in X'_t$ and $g^n(x)$ lies between x and $hf^{+\infty}(x)$ with respect to any geodesic with endpoints in $\{hf^{-\infty}(x), hf^{+\infty}(x)\}$.

We observe that $d(x, f_0(x)) \geq r$, $d(x, g_0(x)) \geq r$ and $(f_0x|f_0^{-1}x)_x \leq s$, $(g_0x|g_0^{-1}x)_x \leq s$, $(f_0^{\pm 1}x|g_0^{\pm 1}x)_x \leq s$. It now follows from Lemma 6.1 that f_0, g_0 freely generate a rank 2 free group that acts essentially freely and ergodically on X'_t . Moreover $\mathcal{T} = \{(x, f_0^{\pm 1}x) : x \in X'_t\} \cup \{(x, g_0^{\pm 1}x) : x \in X'_t\}$ is a treeing of $\mathcal{R} \upharpoonright X'_t$ satisfying the conclusions to this lemma. \square

Lemma 6.3. *Suppose \mathcal{R} is ergodic. Let $X' \subset X$ be a non-null set. If $\mathcal{S}' \leq \mathcal{R} \upharpoonright X'$ is an ergodic treeable subequivalence relation then there exists an ergodic treeable subequivalence relation $\mathcal{S} \leq \mathcal{R}$ with $\mathcal{S} \upharpoonright X' = \mathcal{S}'$. In particular, if \mathcal{S}' is non-hyperfinite then \mathcal{S} is also non-hyperfinite.*

Proof. This Lemma is a special case of [Gab00, Lemme II.8]. For the sake of convenience we provide a proof here. Let $\phi : X \rightarrow X'$ be a measurable map whose graph is contained in \mathcal{R} such that $\phi(x) = x$ for all $x \in X'$. Define \mathcal{S} by $x\mathcal{S}y \Leftrightarrow \phi(x)\mathcal{S}'\phi(y)$. Clearly $\mathcal{S} \upharpoonright X' = \mathcal{S}'$. Since \mathcal{S}' is ergodic and \mathcal{R} is ergodic this implies \mathcal{S} is ergodic. If $\mathcal{T}' \leq \mathcal{S}'$ is a treeing then $\mathcal{T}' \cup \{(x, \phi x), (\phi x, x) : x \in X \setminus X'\}$ is a treeing of \mathcal{S} . Thus \mathcal{S} is treeable. By construction, $\mathcal{S} \upharpoonright X' = \mathcal{S}'$. \square

Proof of Theorem 1.2. The Theorem follows from Lemmas 6.2 and 6.3. \square

7 Parabolic elements

As a consequence of the Tits Alternative, we will show that parabolic elements exist and, in fact, are generic. Using this we prove that the action of the full group $[\mathcal{R}]$ on $\text{Prob}(\mathcal{L} \rightarrow X)$ is minimal.

Theorem 7.1. *Suppose the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite. Then the set of all parabolic elements of $[\mathcal{R}]$, denoted PARA , is a dense G_δ subset of $\text{APER} \subset [\mathcal{R}]$, the set of all aperiodic elements. In particular, there exist ergodic parabolic elements of $[\mathcal{R}]$.*

To begin, we first prove the existence of a single parabolic element.

Lemma 7.2. *Suppose the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite. Then $[\mathcal{R}]$ contains a parabolic element.*

Proof. By Lemma 6.2 there exist a subset $Y \subset X$ with $\mu(Y) > 0$, a subequivalence relation $\mathcal{S} \leq \mathcal{R} \upharpoonright Y$ and a treeing \mathcal{T} of \mathcal{S} such that $\sigma_x(\mathcal{T}_x) \subset \mathcal{H}_x$ is quasi-isometric to a 4-regular metric tree for a.e. x .

By [CMTD14, Lemma 5.3] there exists a one-ended measurable function $f : X \rightarrow X$ whose graph is contained in \mathcal{T} . One-ended means that for a.e. x , if $n \geq 1$ then $f^n x \neq x$ and the backward orbit $\cup_{n=1}^{\infty} f^{-n}x$ is finite. Therefore, the connected components of the graph of f are 1-ended infinite trees. Since $\sigma_x(\mathcal{T}_x)$ is quasi-isometric to a 4-regular metric tree, if $\mathcal{P} \leq \mathcal{S}$ is the subequivalence relation generated by f then the limit set $\mathcal{L}_x(\mathcal{P})$ has cardinality one for a.e. x . In particular, \mathcal{P} is hyperfinite since the map that assigns $x \in X$ to the Dirac measure concentrated on $\mathcal{L}_x(\mathcal{P})$ is a \mathcal{P} -invariant field of probability measures. By Lemmas 3.4 and 3.3 the fact that $\mathcal{L}_x(\mathcal{P})$ has cardinality 1 for a.e. x implies \mathcal{P} is parabolic. Again by Lemma 3.4, it follows that any aperiodic element $g \in [\mathcal{P}]$ is parabolic. \square

Proof of Theorem 7.1. Recall that $\text{Prob}(\mathcal{L} \rightarrow X)$ denotes the space of fields of boundary measures of the bundle $\mathcal{L} \rightarrow X$. By Lemma C.2, $\text{Prob}(\mathcal{L} \rightarrow X)$ is affinely homeomorphic to a compact convex metrizable subset of a Banach space with the weak* topology.

Let $\text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ be the space of all closed subsets of $\text{Prob}(\mathcal{L} \rightarrow X)$ with the Hausdorff topology. Let $\text{Fix} : [\mathcal{R}] \rightarrow \text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ be the map $\text{Fix}(\phi) = \{\nu \in \text{Prob}(\mathcal{L} \rightarrow X) : \phi\nu = \nu\}$. This map is upper semi-continuous in the following sense: if $\{\phi_n\}_{n \in \mathbb{N}} \subset [\mathcal{R}]$ converges to $\phi_\infty \in [\mathcal{R}]$ then

$$\text{Fix}(\phi_\infty) \supset \limsup_{n \rightarrow \infty} \text{Fix}(\phi_n) = \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \text{Fix}(\phi_n)$$

Indeed, this follows from the fact that the map $[\mathcal{R}] \times \text{Prob}(\mathcal{L} \rightarrow X) \rightarrow \text{Prob}(\mathcal{L} \rightarrow X)$ given by $(\phi, \nu) \mapsto \phi\nu$ is continuous (see Theorem C.5).

We say that a subset $\mathcal{K} \subset \text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ is **upwards closed** if $K \in \mathcal{K}$ and $K \subset L \in \text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ implies $L \in \mathcal{K}$. It follows from upper semi-continuity that: if $\mathcal{K} \subset \text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ is closed (in the Hausdorff topology) and upwards closed then $\text{Fix}^{-1}(\mathcal{K}) \subset [\mathcal{R}]$ is closed.

Fix a compatible metric d_* on $\text{Prob}(\mathcal{L} \rightarrow X)$. Given a natural number n , let J_n be the set of all $\nu \in \text{Prob}(\mathcal{L} \rightarrow X)$ such that there exists ν_1, ν_2 with $\nu = (\nu_1 + \nu_2)/2$ and $d_*(\nu_1, \nu_2) \geq 1/n$. Let \mathcal{K}_n be the set of all $L \in \text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ such that $J_n \cap L \neq \emptyset$. Observe that \mathcal{K}_n

is closed in $\text{Closed}(\text{Prob}(\mathcal{L} \rightarrow X))$ and \mathcal{K}_n is upwards closed. So $\text{Fix}^{-1}(\mathcal{K}_n) \subset [\mathcal{R}]$ is closed. By definition,

$$\text{PARA} = \bigcap_{n=1}^{\infty} \text{APER} \setminus \text{Fix}^{-1}(\mathcal{K}_n)$$

is a G_δ subset of APER . By Lemma 7.2, PARA is nonempty. By [Kec10, Theorem 3.4], every conjugacy class in APER is dense. Since parabolicity is a conjugacy-invariant, PARA is a dense G_δ subset of APER .

Since the set $\text{ERG} \subset \text{APER}$ of ergodic elements is a dense G_δ (by [Kec10, Theorem 3.6]) it follows that $\text{PARA} \cap \text{ERG}$ is a dense G_δ subset of APER . In particular, there exists an ergodic parabolic element. \square

We can now prove:

Theorem 1.4 *If the Main Assumption is satisfied and \mathcal{R} is non-hyperfinite then both PARA and LOXO are nonempty. Moreover, PARA is a dense G_δ subset of APER . On the other hand, if \mathcal{R} is hyperfinite then either $\text{APER} = \text{PARA}$ or $\text{APER} = \text{LOXO}$.*

Proof. This first statement follows immediately from Theorems 5.1 and 7.1. To prove the second statement, assume \mathcal{R} is hyperfinite. Since the Main Assumption implies μ is \mathcal{R} -ergodic, we know from Theorem 3.2 that \mathcal{R} is either parabolic or loxodromic. If it is parabolic then every aperiodic subequivalence relation is also parabolic by Theorem 3.5. If it is loxodromic then there exists an \mathcal{R} -invariant field of boundary measures $\nu \in \text{Fix}(\mathcal{R})$ such that the support of ν_x contains two elements for a.e. x . If $\mathcal{S} \leq \mathcal{R}$ is any subequivalence relation then ν is also \mathcal{S} -invariant and therefore, if \mathcal{S} is aperiodic then it must be loxodromic. This proves the last statement. \square

Corollary 7.3. *Suppose the Main Assumption is satisfied. In addition, assume \mathcal{R} is non-hyperfinite. Then the action of $[\mathcal{R}]$ on $\text{Prob}(\mathcal{L} \rightarrow X)$ is minimal.*

Proof. Let $\nu \in \text{Prob}(\mathcal{L} \rightarrow X)$. It suffices to show that the orbit $[\mathcal{R}]\nu$ is dense in $\text{Prob}(\mathcal{L} \rightarrow X)$. By Lemma 7.2 there exists a parabolic element $f \in [\mathcal{R}]$. Since f is parabolic it admits a unique fixed point in $\text{Prob}(\mathcal{L} \rightarrow X)$. Moreover, there is a Borel section $\xi : X \rightarrow \mathcal{L}$ such that the fixed point of f is δ_ξ which denotes the field of Dirac measures $x \mapsto \delta_{\xi(x)}$. Note δ_ξ is an extreme point of $\text{Prob}(\mathcal{L} \rightarrow X)$.

Observe that any limit point of the sequence $n \mapsto \frac{1}{n+1} \sum_{i=0}^n f^i \nu$ must be a fixed point of f and therefore must equal δ_ξ . It follows that δ_ξ is in the closure of the orbit $[\mathcal{R}] \nu$.

Let $\mathcal{O}_x^\xi = \{\alpha(x, y)\xi(y) : y \in [x]_{\mathcal{R}}\}$ be the orbit of ξ and $\mathcal{O}^\xi = \cup_x \mathcal{O}_x^\xi \subset \mathcal{L}$. Because the action of \mathcal{R} on \mathcal{L} is minimal (in the sense of Theorem 2.9) it follows that \mathcal{O}_x^ξ is dense in \mathcal{L}_x for a.e. x .

Fix an arbitrary Borel section $\sigma : X \rightarrow \mathcal{H}$. As in §2 define a fibrewise metric $\rho_\sigma : \overline{\mathcal{H}} * \overline{\mathcal{H}} \rightarrow [0, \infty)$ by

$$\rho_\sigma(\xi, \eta) = \begin{cases} \inf \sum_{i=1}^{n-1} \exp(-\epsilon(\xi_i | \xi_{i+1})_{\sigma(x)}) & \xi \neq \eta \\ 0 & \xi = \eta \end{cases}$$

where $\epsilon > 0$ is such that $\epsilon\delta \leq 1/5$ and the infimum is over all sequences $\xi_1, \dots, \xi_n \in \overline{\mathcal{H}}_x$ with $\xi_1 = \xi, \xi_n = \eta$.

Let $\eta : X \rightarrow \mathcal{L}$ be an arbitrary Borel section and $r > 0$. We will show that δ_η is in the orbit closure of δ_ξ .

Recall from §5.1 that $[\mathcal{R}]$ denotes the partially ordered set of all partial transformations. Let $\mathcal{F} \subset [\mathcal{R}]$ be the set of all ϕ such that

$$\rho_\sigma(\alpha(x, \phi^{-1}x)\xi(\phi^{-1}x), \eta(x)) < r \quad \forall x \in \text{rng}(\phi).$$

By Zorn's Lemma there exists a maximal element ϕ of \mathcal{F} .

To obtain a contradiction, suppose $\phi \notin [\mathcal{R}]$. For $x \notin \text{rng}(\phi)$, let

$$\mathcal{O}_x^{\xi, \phi} = \{\alpha(x, y)\xi(y) : y \in X \setminus \text{dom}(\phi)\}.$$

By Theorem 2.9, $\mathcal{O}_x^{\xi, \phi}$ is dense in \mathcal{L}_x for a.e. x . So there exists a Borel section $\beta : X \setminus \text{rng}(\phi) \rightarrow \mathcal{L}$ such that $\beta(x) \in \mathcal{O}_x^{\xi, \phi}$ for every x and $\rho_\sigma(\beta(x), \eta(x)) < r$. Since $\beta(x) \in \mathcal{O}_x^{\xi, \phi}$ there exists a Borel map $\psi : X \setminus \text{rng}(\phi) \rightarrow X \setminus \text{dom}(\phi)$ with graph contained in \mathcal{R} such that $\alpha(x, \psi(x))\xi(\psi(x)) = \beta(x)$ for a.e. x .

By Lemma B.2, there exists a subset $Y \subset X \setminus \text{rng}(\phi)$ with positive measure such that ψ restricted to Y is 1-1. Thus we map define $\kappa \in [\mathcal{R}]$ by $\kappa(x) = \phi(x)$ for $x \in \text{dom}(\phi)$ and $\kappa(x) = \psi_Y^{-1}(x)$ for $x \in \psi(Y)$ where ψ_Y denotes the restriction of ψ to Y .

Observe that $\kappa \in \mathcal{F}$ by construction. Since $\kappa > \phi$, this contradicts maximality of ϕ . Thus we must have $\phi \in [\mathcal{R}]$.

Note that $\phi\delta_\xi \rightarrow \delta_\eta$ as $r \rightarrow 0$ (ϕ depends implicitly on r). Thus $\delta_\eta \in \overline{[\mathcal{R}]\nu}$. Since η is arbitrary, $\overline{[\mathcal{R}]\nu}$ contains every extreme point of $\text{Prob}(\mathcal{L} \rightarrow X)$ (by Lemma C.3). Because the extreme points of $\text{Prob}(\mathcal{L} \rightarrow X)$ are dense in $\text{Prob}(\mathcal{L} \rightarrow X)$ (Lemma C.4), this implies the corollary. □

A Hyperbolic geometry

Let (\mathcal{H}, d) be a metric space. For $x, y, z \in \mathcal{H}$ define the **Gromov product** of x and y with respect to z by

$$(x|y)_z = (1/2)(d(x, z) + d(y, z) - d(x, y)).$$

A metric space (\mathcal{H}, d) is **δ -hyperbolic** if for every $x, y, z, w \in \mathcal{H}$,

$$(x|y)_z \geq \min\{(x|w)_z, (w|y)_z\} - \delta. \quad (2)$$

A.1 The Gromov boundary

We say that $\{x_i\}_{i=1}^\infty \subset \mathcal{H}$ is a **Gromov sequence** if for some (any) $x \in \mathcal{H}$

$$\lim_{i,j \rightarrow \infty} (x_i|x_j)_x = +\infty.$$

Two Gromov sequence $\{x_i\}_{i=1}^\infty, \{y_i\}_{i=1}^\infty \subset \mathcal{H}$ are **equivalent** if

$$\lim_{i,j \rightarrow \infty} (x_i|y_j)_x = +\infty.$$

Let $\partial\mathcal{H}$ denote the set of all equivalence classes of Gromov sequences. The Gromov product extends to $\overline{\mathcal{H}} := \mathcal{H} \cup \partial\mathcal{H}$ by:

$$(\xi|\eta)_z = \inf \liminf_{i \rightarrow \infty} (1/2)(d(x_i, z) + d(y_i, z) - d(x_i, y_i))$$

where $\xi, \eta \in \partial\mathcal{H}$, $z \in \mathcal{H}$ and the infimum is over all sequences $\{x_i\}_{i=1}^\infty \in \xi, \{y_i\}_{i=1}^\infty \in \eta$. We also define

$$(\xi|y)_z = (y|\xi)_z = \inf \liminf_{i \rightarrow \infty} (1/2)(d(x_i, z) + d(y, z) - d(x_i, y))$$

where $\xi \in \partial\mathcal{H}, y, z \in \mathcal{H}$ and the infimum is over all sequences $\{x_i\}_{i=1}^\infty \in \xi$. An elementary computation shows that at the cost of increasing δ , the equation (2) holds for all $x, y, w \in \overline{\mathcal{H}}$

and $z \in \mathcal{H}$. We will therefore assume that $\delta > 0$ has been chosen so that equation (2) holds for all $x, y, w \in \overline{\mathcal{H}}$ and $z \in \mathcal{H}$. For $\epsilon > 0$ and $\xi, \eta \in \overline{\mathcal{H}}$, define

$$\rho_\epsilon(\xi, \eta) = \begin{cases} \inf \sum_{i=1}^{n-1} \exp(-\epsilon(\xi_i | \xi_{i+1})_x) & \xi \neq \eta \\ 0 & \xi = \eta \end{cases}$$

where the infimum is over all sequences $\xi_1, \dots, \xi_n \in \overline{\mathcal{H}}$ with $\xi_1 = \xi, \xi_n = \eta$.

Lemma A.1. *If $\epsilon\delta \leq 1/5$ then ρ_ϵ is a metric on $\overline{\mathcal{H}}$.*

Proof. This is implied by [Väi05, Proposition 5.16]. □

Lemma A.2. *If d is a complete metric on \mathcal{H} then ρ_ϵ is a complete metric on $\overline{\mathcal{H}}$ ($\epsilon\delta \leq 1/5$). If d is also proper and geodesic then $\overline{\mathcal{H}}$ is compact.*

Proof. The first statement is [Väi05, Proposition 5.31]. The second statement is [BH99, Chapter III.H, Proposition 3.7]. □

Definition 15 (Geodesics). A path $\gamma : I \rightarrow \mathcal{H}$ is a **geodesic** if $d(\gamma(t), \gamma(s)) = |t - s|$ for any $t, s \in I$ where $I \subset \mathbb{R}$ is an interval (possibly an infinite interval). If $I = [a, b]$ then the **endpoints of γ** are $\gamma(a), \gamma(b)$. If $I = (-\infty, +\infty)$ then the **endpoints of γ** are $\lim_{t \rightarrow \pm\infty} \gamma(t)$. If $p, q \in \overline{\mathcal{H}}$ then we may write $[p, q]$ to denote a geodesic with endpoints p, q . This geodesic might not be unique. By abuse of notation, we may identify this geodesic with its image as a subset of \mathcal{H} . We say (\mathcal{H}, d) is a **geodesic metric space** if for every $a, b \in \mathcal{H}$ there is a geodesic with endpoints $\{a, b\}$.

Lemma A.3 (Thin triangles). *Suppose (\mathcal{H}, d) is a complete geodesic hyperbolic metric space and $\gamma_1, \gamma_2, \gamma_3$ are geodesics forming a triangle (so if γ_i has endpoints p_i, q_i then $q_i = p_{i+1} \bmod 3$). Then there is a $\delta' > 0$ such that the δ' -neighborhood of $\gamma_1 \cup \gamma_2$ contains γ_3 . In particular, if γ_1, γ_2 are geodesics with the same endpoints then γ_2 is contained in the $2\delta'$ -neighborhood of γ_1 .*

Proof. This is [BH99, Chapter III.H, Proposition 1.22]. □

A.2 Quasi-isometries

Definition 16. Let (X, d_X) , (Y, d_Y) be metric spaces. For $\lambda \geq 1$ and $c \geq 0$, a map $\phi : X \rightarrow Y$ is a (λ, c) -**quasi-isometric embedding** if for all $x, y \in X$,

$$\lambda^{-1}d_X(x, y) - c \leq d_Y(\phi(x), \phi(y)) \leq \lambda d_X(x, y) + c.$$

Definition 17. Let (\mathcal{H}, d) be a Gromov hyperbolic space. A (λ, c) -quasi-isometric embedding q of an interval $I \subset \mathbb{R}$ into \mathcal{H} is called a (λ, c) -**quasi-geodesic**. A **quasi-geodesic** is a (λ, c) -quasi-geodesic for some λ, c . If I is a finite interval and its endpoints are mapped to $x, y \in \mathcal{H}$ respectively, then we say it is a quasi-geodesic **from x to y** . If $I = (-\infty, \infty)$ and $\lim_{t \rightarrow -\infty} q(t) = \xi_-, \lim_{t \rightarrow +\infty} q(t) = \xi_+$, then we say q is a quasi-geodesic from ξ_- to ξ_+ . A similar definition holds for half-infinite intervals.

A.3 Busemann functions

Let (\mathcal{H}, d) be a δ -hyperbolic metric space. Given $\xi \in \partial\mathcal{H}$, the associated **Busemann function** $\beta : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ is defined by

$$\beta(x, y) = \sup \limsup_{i \rightarrow \infty} d(x, z_i) - d(y, z_i) = \sup \limsup_{i \rightarrow \infty} 2(y|z_i)_x - d(x, y)$$

where the supremum is over all sequences $\{z_i\}_{i \in \mathbb{N}}$ of points $z_i \in \mathcal{H}$ such that $\lim_{i \rightarrow \infty} z_i = \xi$.

Lemma A.4. *Suppose $\{p_i\} \subset \mathcal{H}$ is a sequence converging to ξ in $\overline{\mathcal{H}}$. Then for any $x, y \in \mathcal{H}$,*

$$|\beta(x, y) - (\liminf_{i \rightarrow \infty} d(x, p_i) - d(y, p_i))| \leq 2\delta$$

and

$$|\beta(x, y) - (\limsup_{i \rightarrow \infty} d(x, p_i) - d(y, p_i))| \leq 2\delta.$$

Proof. By definition of β , it suffices to show

$$\limsup_{i, j \rightarrow \infty} |2(y|z_i)_x - 2(y|p_j)_x| \leq 2\delta.$$

By (2), $(y|z_i)_x \geq \min\{(y|p_j)_x, (p_j|z_i)_x\} - \delta$. Since $(p_j|z_i)_x \rightarrow \infty$ as $i, j \rightarrow \infty$, this implies that for all i, j sufficiently large, $(y|z_i)_x \geq (y|p_j)_x - \delta$. By symmetry, this implies the claim above. \square

Lemma A.5. *Suppose x, y, ξ all lie on a geodesic γ . Then $|\beta(x, y) - d(x, y)| \leq 2\delta$.*

Proof. This follows from the previous lemma by choosing the points p_i to lie on the geodesic γ . □

Lemma A.6. *For any $\xi \in \partial\mathcal{H}$ and $x, y, z \in \mathcal{H}$ the cocycle equation holds up to 4δ :*

$$|\beta(x, y) + \beta(y, z) - \beta(x, z)| \leq 4\delta.$$

Proof. Choose a sequence $\{p_i\} \subset \mathcal{H}$ converging to ξ such that

$$\beta(x, y) = \lim_{i \rightarrow \infty} d(x, p_i) - d(y, p_i).$$

Lemma A.4 implies that, up to a maximum error of 4δ , $\beta(x, y) + \beta(y, z) - \beta(x, z)$ equals

$$\limsup_i d(x, p_i) - d(y, p_i) + d(y, p_i) - d(z, p_i) - (d(x, p_i) - d(z, p_i)) = 0.$$

□

B Measured equivalence relations

Let (X, μ) be a standard Borel probability space and $\mathcal{R} \subset X \times X$ be a Borel equivalence relation. We say that \mathcal{R} is

- **discrete** if every \mathcal{R} -class is countable,
- **probability-measure-preserving** (pmp) if for every Borel isomorphism $\phi : X \rightarrow X$ with $x\mathcal{R}\phi(x)$ for all x , we have $\phi_*\mu = \mu$,
- **ergodic** if for every Borel set $A \subset X$, $\mu([A]_{\mathcal{R}}) \in \{0, 1\}$ where $[A]_{\mathcal{R}}$ is the union of all \mathcal{R} -classes that nontrivially intersect A .

Two discrete pmp equivalence relations $(X_i, \mu_i, \mathcal{R}_i)$ (for $i = 1, 2$) are **isomorphic** if there exist conull sets $X'_i \subset X_i$ and a measure-space isomorphism $\phi : (X'_1, \mu_1) \rightarrow (X'_2, \mu_2)$ such that $(x, y) \in \mathcal{R}_1 \Leftrightarrow (\phi(x), \phi(y)) \in \mathcal{R}_2$. More precisely, we only require that ϕ is defined on a set of full measure.

Define measures μ_L, μ_R on \mathcal{R} by

$$\mu_L(F) = \int |F \cap \pi_L^{-1}(x)| d\mu(x), \quad \mu_R(F) = \int |F \cap \pi_R^{-1}(x)| d\mu(x)$$

where $\pi_L : \mathcal{R} \rightarrow X, \pi_R : \mathcal{R} \rightarrow X$ are the left and right projection maps. It is a standard fact (and a good exercise) to show that \mathcal{R} is pmp if and only if $\mu_L = \mu_R$. In this case, we let $\hat{\mu}$ denote either μ_L or μ_R . In the sequel, the phrase “for a.e. $(x, y) \in \mathcal{R}$ ” is taken to mean with respect to $\hat{\mu}$. Another formulation of the probability measure-preserving property is the following:

Lemma B.1 (Mass Transport Principle). *If \mathcal{R} is a discrete pmp Borel equivalence relation on (X, μ) and suppose either $F \in L^1(\mathcal{R}, \hat{\mu})$ or $F \geq 0$ is measurable. Then*

$$\int \sum_{x \in [y]_{\mathcal{R}}} F(x, y) d\mu(y) = \int \sum_{y \in [x]_{\mathcal{R}}} F(x, y) d\mu(x) = \int F d\hat{\mu}.$$

Proof. Apply Fubini’s Theorem. □

Measured equivalence relations arise from actions of groups: if G is a countable group and $G \curvearrowright (X, \mu)$ a measure-class-preserving action then $\mathcal{R} = \{(x, gx) : x \in X, g \in G\}$ is a discrete Borel equivalence relation called the **orbit equivalence relation**. This action of G is measure-preserving if and only if \mathcal{R} is pmp; the action of G is ergodic if and only if μ is \mathcal{R} -ergodic. Feldman-Moore [FM77] proved that every discrete pmp equivalence relation \mathcal{R} is the orbit-equivalence relation of the action of some countable group. One of the useful consequences of this is:

Lemma B.2. *Let \mathcal{R} be a discrete pmp equivalence relation, $Y \subset X$ Borel and $\phi : Y \rightarrow X$ a Borel map with $x\mathcal{R}\phi(x)$ for all $x \in Y$. If $\mu(Y) > 0$ then there exists $Z \subset Y$ with $\mu(Z) > 0$ such that ϕ restricted to Z is 1-1.*

Proof. As mentioned above, we may assume \mathcal{R} is generated by a countable group G . For $g \in G$, let $Y_g = \{x \in Y : \phi x = gx\}$. Since $Y = \cup_g Y_g$, there exists $g \in G$ such that $\mu(Y_g) > 0$. Since ϕ restricted to Y_g is 1-1, we are done. □

B.1 The full group

The full group of \mathcal{R} , denoted $[\mathcal{R}]$, is the group of all (equivalence classes of) invertible Borel transformations f such that $\text{graph}(f) = \{(x, fx) : x \in X\} \subset \mathcal{R}$. Two transformations are equivalent if they agree on a conull subset. By [Kec10, Proposition 3.2], $[\mathcal{R}]$ with the uniform metric, defined by

$$d_u(\phi, \psi) = \mu(\{x \in X : \phi(x) \neq \psi(x)\}),$$

is a Polish group.

An element $\phi \in [\mathcal{R}]$ is

- **aperiodic** if for a.e. $x \in X$, $\{\phi^n(x)\}_{n \in \mathbb{Z}}$ is infinite
- **ergodic** if every measurable ϕ -invariant subset $A \subset X$ satisfies $\mu(A) \in \{0, 1\}$.

Since ϕ is aperiodic if and only if $d_u(\phi^n, \text{id}) > 1 - 1/m$ for every $n, m \geq 1$, the subset $\text{APER}(\mathcal{R})$ of aperiodic elements of $[\mathcal{R}]$ is a G_δ subset. So it is Polish. By [Kec10, Theorem 3.6] the subset $\text{ERG}(\mathcal{R}) \subset \text{APER}(\mathcal{R})$ of ergodic elements is a dense G_δ subset of $\text{APER}(\mathcal{R})$.

B.2 Subequivalence relations

A Borel subset $\mathcal{S} \subset X \times X$ is a **subequivalence relation** of \mathcal{R} if it is an equivalence relation and $\mathcal{S} \subset \mathcal{R}$. This is denoted by $\mathcal{S} \leq \mathcal{R}$. For example if \mathcal{R} is the orbit-equivalence relation of the action $G \curvearrowright (X, \mu)$ of a countable group G and $H < G$ is a subgroup then orbit-equivalence relation of H is a subequivalence relation of \mathcal{R} .

B.3 Compressions and amplifications

Let (X, μ, \mathcal{R}) be a pmp equivalence relation and $Y \subset X$ a measurable subset with $\mu(Y) > 0$. Then $\mathcal{R} \upharpoonright Y := \mathcal{R} \cap (Y \times Y)$ is an equivalence relation on Y called the **compression (or restriction) of \mathcal{R} to Y** . It preserves the restricted measure $\mu \upharpoonright Y$. Moreover, if (\mathcal{R}, μ) is ergodic/treeable/hyperfinite then $(\mathcal{R} \upharpoonright Y, \mu \upharpoonright Y)$ is also ergodic/treeable/hyperfinite. If (\mathcal{R}, μ) satisfies the Main Assumption (Definition 2) then $(\mathcal{R} \upharpoonright Y, \mu \upharpoonright Y)$ also satisfies the Main Assumption by restricting the bundle \mathcal{H} to $\pi^{-1}(Y)$.

Let F denote a finite or countably infinite set. We define an equivalence relation $\tilde{\mathcal{R}}$ on $X \times F$ by $(x, n)\tilde{\mathcal{R}}(y, m) \Leftrightarrow x\mathcal{R}y$. Fixing an element $f_0 \in F$, we may identify X with $X \times \{f_0\}$ and \mathcal{R} with the compression $\tilde{\mathcal{R}} \upharpoonright X \times \{f_0\}$. We call $\tilde{\mathcal{R}}$ the **amplification** of \mathcal{R} to $X \times F$. It is measure-preserving if we give $X \times F$ the measure $\mu \times c_F$ where c_F denotes counting measure on F . Again it is elementary to check that if (\mathcal{R}, μ) is ergodic/treeable/hyperfinite then $(\tilde{\mathcal{R}}, \mu \times c_F)$ is ergodic/treeable/hyperfinite. If (\mathcal{R}, μ) satisfies the Main Assumption then $(\tilde{\mathcal{R}}, \mu \times c_F)$ does too: we simply let $\mathcal{H} \times F \rightarrow X \times F$ denote the obvious extension of the bundle $\mathcal{H} \rightarrow X$. We leave the details to the reader.

B.4 Finite equivalence relations

Definition 18. Let (X, μ, \mathcal{R}) be a pmp discrete equivalence relation. A **selector** is a measurable map $f : X \rightarrow X$ such that $f(x)\mathcal{R}x$ for a.e. x and $f(x) = f(y)$ for a.e. $(x, y) \in \mathcal{R}$. A **set function** for \mathcal{R} is a map F on X such that $F(x)$ is a subset of $[x]_{\mathcal{R}}$. We require that F is measurable which means that the subset $\{(x, y) \in \mathcal{R} : y \in F(x)\} \subset \mathcal{R}$ is measurable. A set function is **finite** if $F(x)$ is finite for a.e. x . It is **invariant** if $F(x) = F(y)$ for a.e. $(x, y) \in \mathcal{R}$.

Lemma B.3. *Let (X, μ, \mathcal{R}) be a pmp discrete equivalence relation. The following are equivalent:*

- *There is a finite invariant set function for \mathcal{R} ;*
- *There is a selector for \mathcal{R} ;*
- *for a.e. $x \in \mathcal{R}$, $[x]_{\mathcal{R}}$ is finite.*

Proof. The lemma is trivial in the atomic case so without loss of generality we may assume μ is purely non-atomic. So there exists a Borel isomorphism $\phi : X \rightarrow [0, 1]$. Suppose F is a finite invariant set function for \mathcal{R} . Define $f : X \rightarrow X$ by

$$f(x) = y \Leftrightarrow y \in F(x) \text{ and } \phi(y) = \min\{\phi(z) : z \in F(x)\}.$$

Then f is a selector for \mathcal{R} . Observe that $\phi \circ f : X \rightarrow [0, 1]$ is an \mathcal{R} -invariant function satisfying $x\mathcal{R}y \Leftrightarrow \phi(f(x)) = \phi(f(y))$ for a.e. $(x, y) \in \mathcal{R}$. In particular, if \mathcal{R} is ergodic then

$\phi \circ f$ must be constant and therefore there must be only one \mathcal{R} -class (up to measure zero). Since μ is an invariant probability measure, this implies that $[x]_{\mathcal{R}}$ is finite for a.e. $x \in \mathcal{R}$. The general case follows from the ergodic decomposition theorem.

If $[x]_{\mathcal{R}}$ is finite for a.e. x then the set function $F(x) = [x]_{\mathcal{R}}$ is a finite invariant set function. \square

C Fields of probability measures

Throughout this section, we let $\pi : B \rightarrow X$ and $d : B * B \rightarrow \mathbb{R}$ denote a separable metric bundle (as in Definition 1). We also fix a Borel probability measure μ on X . A **field of probability measures** is an assignment $x \mapsto \nu_x$ of probability measures on B_x (for $x \in X$). We will topologize the space of Borel fields of probability measures and prove that it is Hausdorff, separable and even compact under appropriate hypotheses. We will then investigate its convex structure.

To begin, we need a little notation. We say two functions F_1, F_2 on B are **equivalent** if for a.e. $x \in X$ $F_1 \upharpoonright B_x = F_2 \upharpoonright B_x$ where \upharpoonright means “restricted to”. For each $x \in X$, let $C_0(B_x)$ denote the Banach space of continuous functions on B_x that vanish at infinity with the sup norm. Suppose $F : B \rightarrow \mathbb{C}$ is a Borel function such that $F_x \in C_0(B_x)$ for a.e. $x \in X$. Then we define its norm by

$$\|F\| := \|x \mapsto \|F_x\|\|_{L^\infty(X, \mu)}.$$

Let $C_0(\pi)$ denote the set of all equivalence classes of Borel functions $F : B \rightarrow \mathbb{C}$ such that for a.e. $x \in X$, $F_x \in C_0(B_x)$ and $\|F\| < \infty$.

Lemma C.1. *There exists a countable set $\Omega \subset C_0(\pi)$ such that for a.e. $x \in X$, $\{F_x\}_{F \in \Omega}$ is dense in $C_0(B_x)$.*

Proof. Because the bundle is separable there exists a sequence $\{\sigma_i\}_{i \in \mathbb{N}}$ of Borel sections and $Y \subset X$ a conull set such that for every $x \in Y$, $\{\sigma_i(x)\}_{i \in \mathbb{N}}$ is dense in B_x . For $n, i \in \mathbb{N}$ define $F_{n,i} \in C_0(\pi)$ as follows. For $p \in B$ with $\pi(p) = x$ let

$$F_{n,i}(p) = \max(1/n - d(\sigma_i(x), p), 0).$$

Let \mathbb{F} be a countable dense subfield of \mathbb{C} . Let Ω be the \mathbb{F} -subalgebra of $C_0(\pi)$ generated by $\{F_{n,i}\}_{n,i \in \mathbb{N}}$. By the Stone-Weierstrauss Theorem, for every $x \in Y$, the \mathbb{C} -linear span of $\{F_x\}_{F \in \Omega}$ is dense in $C_0(B_x)$. Since \mathbb{F} is dense in \mathbb{C} , we obtain that in fact $\{F_x\}_{F \in \Omega}$ is dense in $C_0(B_x)$. \square

Let $\mathcal{P}(B_x)$ denote the set of all Borel probability measures on B_x and let $\mathcal{P}(B) = \sqcup_x \mathcal{P}(B_x)$ denote the disjoint union. This is a bundle of spaces of probability measures over X . A **Borel field of regular probability measures** is a map $\nu : X \rightarrow \mathcal{P}(B)$ satisfying:

- for every $x \in X$, ν_x is a regular Borel probability measure on the fiber B_x
- for every $F \in \mathcal{F}(B)$, the map $x \mapsto \int F(p) d\nu_x(p)$ is measurable.

Two fields ν, η are **equivalent** if $\nu_x = \eta_x$ for a.e. x . By abusing notation, we will not distinguish between equivalent fields.

Let $\text{Prob}(\pi)$ denote the set of all (equivalence classes of) Borel fields of regular probability measures on B . Given $F \in C_0(\pi)$ and an open set $O \subset \mathbb{C}$, let $\Omega(F, O)$ be the set of all $\nu \in \text{Prob}(\pi)$ such that $\int \nu_x(F_x) d\mu(x) \in O$. We give $\text{Prob}(\pi)$ the topology generated by sets of the form $\Omega(F, O)$.

Let $C_0(\pi)^*$ denote the Banach dual of $C_0(\pi)$. We always consider $C_0(\pi)^*$ with the weak* topology. This means that $\Lambda_i \rightarrow \Lambda$ in $C_0(\pi)^*$ if and only if $\Lambda_i(F) \rightarrow \Lambda(F)$ for every $F \in C_0(\pi)$. Let $\Psi : \text{Prob}(\pi) \rightarrow C_0(\pi)^*$ denote the map $\Psi(\nu)(F) = \int \nu_x(F_x) d\mu(x)$.

Lemma C.2. *The map Ψ is an affine homeomorphism onto its image. Thus $\text{Prob}(\pi)$ is metrizable and convex. If B_x is compact (for a.e. x) then $\text{Prob}(\pi)$ is compact.*

Proof. It is easy to check that Ψ is affine (this means that $\Psi(t\nu + (1-t)\eta) = t\Psi(\nu) + (1-t)\Psi(\eta)$ for any ν, η and $t \in [0, 1]$) and continuous. To see that it is injective, let $\nu, \eta \in \text{Prob}(\pi)$ and suppose $\Psi(\nu) = \Psi(\eta)$. So for every $F \in C_0(\pi)$ we have

$$\int \nu_x(F_x) d\mu(x) = \int \eta_x(F_x) d\mu(x).$$

We claim that $\nu_x(F_x) = \eta_x(F_x)$ for a.e. x . Indeed, for any Borel set $E \subset X$, $\chi_{\pi^{-1}E}F \in C_0(\pi)$ where $\chi_{\pi^{-1}E}$ denotes the characteristic function of $\pi^{-1}E \subset B$. So

$$\int_E \nu_x(F_x) d\mu(x) = \int \nu_x((\chi_{\pi^{-1}E}F)_x) d\mu(x) = \int \eta_x((\chi_{\pi^{-1}E}F)_x) d\mu(x) = \int_E \eta_x(F_x) d\mu(x).$$

Since $E \subset X$ is arbitrary, for every $F \in C_0(\pi)$ we have $\nu_x(F_x) = \eta_x(F_x)$ for a.e. x .

Let $\Omega \subset C_0(\pi)$ be as in Lemma C.1. Because Ω is countable, there is a conull set $Z \subset X$ such that $\nu_x(F_x) = \eta_x(F_x)$ for every $F \in \Omega$ and $x \in Z$. Because ν_x and η_x are regular, they are determined by their values on $C_0(B_x)$. Therefore $\eta_x = \nu_x$ for all $x \in Z$. Since Z is conull, this proves Ψ is injective.

It is easy to check that the inverse Ψ^{-1} is also continuous and therefore Ψ is an affine homeomorphism onto its image. We define a metric on $\text{Prob}(\pi)$ as follows. Let $\Omega = \{F_i\}_{i \in \mathbb{N}}$ and define

$$d_{\text{Prob}(\pi)}(\nu, \eta) := \sum_{i \in \mathbb{N}} \frac{|\Psi(\nu)(F_i) - \Psi(\eta)(F_i)|}{\|F_i\| 2^i}.$$

The image of Ψ lies inside the unit ball of $C_0(\pi)^*$ which, by the Banach-Alaoglu Theorem, is weak* compact. Now suppose each fiber B_x is compact. It suffices to show the image of Ψ is weak* closed. So suppose $\eta^j \in \text{Prob}(\pi)$ and $\Psi(\eta^j) \rightarrow \Lambda \in C_0(\pi)^*$ as $j \rightarrow \infty$. Let $F \in C_0(\pi)$ and $E \subset X$ be Borel. Since

$$\lim_i \int \eta_x^j((\chi_{\pi^{-1}E}F)_x) d\mu(x) = \lim_i \int_E \eta_x^j(F_x) d\mu(x) = \Lambda(\chi_{\pi^{-1}E}F)$$

it follows that the function $E \mapsto \Lambda(\chi_{\pi^{-1}E}F)$ is a complex valued measure on X that is absolutely continuous μ . By the Radon-Nikodym Theorem there exists a function $\rho_F : X \rightarrow \mathbb{C}$ such that

$$\int_E \rho_F d\mu = \Lambda(\chi_{\pi^{-1}E}F)$$

for Borel $E \subset X$. Note also that the functions $x \mapsto \eta_x^j(F_x)$ converge in measure to ρ_F . So after passing to a subsequence if necessary, we may assume that $\eta_x^j(F_x)$ converges to $\rho_F(x)$ pointwise a.e. as $j \rightarrow \infty$ and for every $F \in \Omega$. However this implies that $\{\eta_x^j\}$ converges in the weak* topology on $C_0(B_x)^*$ as $j \rightarrow \infty$ (for a.e. x) (since $\{F_x\}_{F \in \Omega}$ is dense in $C_0(B_x)$). Moreover, since each B_x is compact, the limiting measure, denoted λ_x , is a Borel probability measure. Thus we have obtained probability measures λ_x such that $\eta_x^j(F_x) \rightarrow \lambda_x(F_x) = \rho_F(x)$ for a.e. x . In particular the field λ is in $\text{Prob}(\pi)$ and $\int \lambda_x(F_x) d\mu(x) = \Lambda(F)$ for every $F \in C_0(\pi)$ which, by injectivity, implies that $\Psi(\lambda) = \Lambda$. This proves that the image of Ψ is closed and therefore $\text{Prob}(\pi)$ is compact, as required. □

Recall that if \mathcal{C} is a convex subspace of a Banach space then a point $x \in \mathcal{C}$ is **extreme** if and only if there does not exist element $y, z \in \mathcal{C}$ and $t \in (0, 1)$ such that $y \neq z$ and $x = ty + (1 - t)z$. A **Dirac measure** is a probability measure whose support contains only one element.

Lemma C.3. *Assume that B is a separable bundle and for each $x \in X$, B_x is compact and Hausdorff. Then $\beta \in \text{Prob}(\pi)$ is extreme if and only if for a.e. x , β_x is a Dirac measure on B_x .*

Proof. Clearly, if $\beta \in \text{Prob}(\pi)$ is such that $\beta(x)$ is a Dirac measure for a.e. x then β is extreme.

On the other hand suppose $\nu \in \text{Prob}(\pi)$ and if Y is the set of all $x \in X$ such that ν_x is not a Dirac measure then $\mu(Y) > 0$. It suffices to show ν is not extreme.

Let $\text{supp}(\nu_x)$ denote the support of ν_x and $\text{diam}(\text{supp}(\nu_x))$ its diameter. Because ν_x is not a Dirac measure for $x \in Y$, there exists a number $r > 0$ such that if $Y_r = \{y \in Y : \text{diam}(\text{supp}(\nu_y)) > r\}$ then $\mu(Y_r) > 0$.

Because the bundle $\pi : B \rightarrow X$ is separable there exists a Borel section $\sigma : X \rightarrow B$ such that

$$d(\sigma(x), \text{supp}(\nu_x)) \leq r/10$$

for a.e. $x \in Y_r$. Let $N_{r/3}(\sigma(x))$ denote the closed $r/3$ -neighborhood of $\sigma(x)$ in B_x . If $x \in Y_r$ then $N_{r/3}(\sigma(x))$ contains a nonempty open subset of $\text{supp}(\nu_x)$ and its complement also contains a nontrivial open subset of $\text{supp}(\nu_x)$. Therefore

$$0 < \nu_x(N_{r/3}(\sigma(x))) < 1.$$

So there exists a $0 < t < 1$ such that if $Z_t = \{x \in Y_r : t < \nu_x(N_{r/3}(\sigma(x))) < \frac{1}{1+t}\}$ then $\mu(Z_t) > 0$.

Given any subset $C \subset B_x$ with $\nu_x(C) > 0$, let $\nu_x \upharpoonright C$ denote the probability measure obtained by restricting ν_x to C and normalizing so that $\nu_x \upharpoonright C$ is a probability measure.

Define $\nu^1, \nu^2 \in \text{Prob}(\pi)$ as follows. For $y \notin Z_t$, let $\nu_y^1 = \nu_y^2 = \nu_y$.

For $x \in Y_r$, let

$$\nu_x^1 = (1 + t)\nu_x(N_{r/3}(\sigma x)) [\nu_x \upharpoonright N_{r/3}(\sigma x)] + (1 - (1 + t)\nu_x(N_{r/3}(\sigma x))) [\nu_x \upharpoonright (B_x \setminus N_{r/3}(\sigma x))]$$

$$\nu_x^2 = (1-t)\nu_x(N_{r/3}(\sigma x)) [\nu_x \upharpoonright N_{r/3}(\sigma x)] + (1-(1-t)\nu_x(N_{r/3}(\sigma x))) [\nu_x \upharpoonright (B_x \setminus N_{r/3}(\sigma x))].$$

Observe that $\nu^1 \neq \nu^2$ and yet

$$\nu = \frac{\nu^1 + \nu^2}{2}.$$

So ν is not extremal. □

Lemma C.4. *Let $\text{Prob}^{ex}(\pi) \subset \text{Prob}(\pi)$ denote the subspace of extreme points. Then $\text{Prob}^{ex}(\pi)$ is a dense G_δ subset of $\text{Prob}(\pi)$.*

Proof. By Lemma C.2 there exists a metric $d_{\text{Prob}(\pi)}$ on $\text{Prob}(\pi)$. For $n \in \mathbb{N}$, let F_n denote the set of all $\nu \in \text{Prob}(\pi)$ such that there exist $\nu^1, \nu^2 \in \text{Prob}(\pi)$ such that $d_{\text{Prob}(\pi)}(\nu^1, \nu^2) \geq 1/n$ and $\nu = \frac{\nu^1 + \nu^2}{2}$. Then F_n is closed in $\text{Prob}(\pi)$ and $\text{Prob}^{ex}(\pi)$ is the complement of $\bigcup_{n \in \mathbb{N}} F_n$. This proves $\text{Prob}^{ex}(\pi)$ is a G_δ .

To prove that $\text{Prob}^{ex}(\pi)$ is dense in $\text{Prob}(\pi)$ it suffices to show that the closure of $\text{Prob}^{ex}(\pi)$ is convex. So let ν, η be in the closure of $\text{Prob}^{ex}(\pi)$. It suffices to show $(1/2)(\nu + \eta)$ is in the closure of $\text{Prob}^{ex}(\pi)$.

Since ν, η are in the closure, there exist extremal measures ν^i, η^i with $\nu^i \rightarrow \nu$ and $\eta^i \rightarrow \eta$ in $\text{Prob}(\pi)$. Since $(1/2)(\nu^i + \eta^i)$ converges to $(1/2)(\nu + \eta)$ it suffices to show that $(1/2)(\nu^i + \eta^i)$ is in the closure of $\text{Prob}^{ex}(\pi)$. So we have reduced the problem to showing: if $\nu, \eta \in \text{Prob}^{ex}(\pi)$ then $(1/2)(\nu + \eta)$ is in the closure of $\text{Prob}^{ex}(\pi)$.

Because (X, μ) is a standard nonatomic probability space, there exists a sequence $\{E_n\}$ of measurable sets $E_n \subset X$ such that $\mu(E_n) = 1/2$ for all n and if $A \subset X$ is any measurable subset then

$$\lim_{n \rightarrow \infty} \mu(E_n \cap A) = \mu(A)/2.$$

Define $\beta^n \in \text{Prob}(\pi)$ by $\beta_x^n = \nu_x$ if $x \in E_n$ and $\beta_x^n = \eta_x$ if $x \notin E_n$. Since β_x^n is Dirac for a.e. x , $\beta^n \in \text{Prob}^{ex}(\pi)$. It is an easy exercise to show $\beta^n \rightarrow (1/2)(\nu + \eta)$ as $n \rightarrow \infty$ (with respect to the weak* topology on $\text{Prob}(\pi)$). □

The reader is cautioned here that $\text{Prob}(\pi)$ is not in general a simplex (unless μ is a Dirac measure). That is, it is not necessarily true that every $\nu \in \text{Prob}(\pi)$ can be decomposed as a convex integral of extreme points *uniquely*. For example, suppose (C, d_C) is a metric

space and $B = X \times C$, $\pi : B \rightarrow X$ is the usual projection map and $d : B * B \rightarrow \mathbb{R}$ is the metric $d((x, c), (x, c')) = d_C(c, c')$. In this case, $\text{Prob}(\pi)$ can be identified with the set of (a.e. equivalence classes of) Borel maps from X into $\text{Prob}(C)$, the space of regular Borel probability measures on C . If $\sigma_1, \sigma_2 : X \rightarrow C$ are two Borel maps such that $\sigma_1(x) \neq \sigma_2(x)$ for a.e. x , $Y \subset X$ has measure $1/2$ and $\nu_x^i = \delta_{\sigma_i(x)}$ for $x \in Y$, $\nu_x^i = \delta_{\sigma_{i+1}(x)}$ for $x \notin Y$ (indices mod 2) then ν^1, ν^2 are both extreme and

$$(1/2)(\nu^1 + \nu^2) = (1/2)(\delta_{\sigma_1} + \delta_{\sigma_2})$$

are two different extremal decompositions of the same Borel field of probability measures. So $\text{Prob}(\pi)$ is not a simplex.

Now suppose $\mathcal{R} \subset X \times X$ is a discrete Borel equivalence relation on X and $\{\alpha(x, y) : (x, y) \in \mathcal{R}\}$ is a Borel action on B by homeomorphisms. To be precise, this means that:

- $\alpha(x, y) : B_y \rightarrow B_x$ is a homeomorphism;
- $\{(p, q) \in B \times B : \alpha(\pi(p), \pi(q))(q) = p\}$ is Borel.

Let $[\mathcal{R}]$ denote the full group of \mathcal{R} with the uniform topology (see §B.1). This group acts on $\text{Prob}(\pi)$ by $(f\nu)_{fx} = \alpha(fx, x)_* \nu_x$.

Theorem C.5. *The action of $[\mathcal{R}]$ on $\text{Prob}(\pi)$ is jointly continuous.*

Proof. Let $\{f_i\}_{i \in \mathbb{N}} \subset [\mathcal{R}]$ converge to $f \in [\mathcal{R}]$ and $\{\eta^i\}_{i \in \mathbb{N}} \subset \text{Prob}(\pi)$ converge to $\eta \in \text{Prob}(\pi)$. Let $F \in C_0(\pi)$. Observe:

$$\begin{aligned} \lim_{i \rightarrow \infty} \int (f_i \eta^i)_x(F_x) d\mu(x) &= \lim_{i \rightarrow \infty} \int (\alpha(x, f_i^{-1}x)_* \eta_{f_i^{-1}x}^i)(F_x) d\mu(x) \\ &= \lim_{i \rightarrow \infty} \int (\alpha(f_i x, x)_* \eta_x^i)(F_{f_i x}) d\mu(x) \\ &= \lim_{i \rightarrow \infty} \int \eta_x^i(F_{f_i x} \circ \alpha(f_i x, x)) d\mu(x). \end{aligned}$$

Because $f_i \rightarrow f$ in uniformly

$$\lim_{i \rightarrow \infty} \mu(\{x \in X : F_{f_i x} \circ \alpha(f_i x, x) = F_{fx} \circ \alpha(fx, x)\}) = 1.$$

Since F is essentially bounded the previous limit equals

$$= \lim_{i \rightarrow \infty} \int \eta_x^i(F_{fx} \circ \alpha(fx, x)) d\mu(x).$$

Since $\eta^i \rightarrow \eta$ as $i \rightarrow \infty$ we have

$$= \lim_{i \rightarrow \infty} \int \eta_x(F_{fx} \circ \alpha(fx, x)) \, d\mu(x) = \lim_{i \rightarrow \infty} \int (f\eta)_x(F_x) \, d\mu(x).$$

Because F is arbitrary, this proves $\lim_{i \rightarrow \infty} f_i \eta^i = f\eta$. So $[\mathcal{R}] \curvearrowright \text{Prob}(\pi)$ is jointly continuous. \square

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